Kernelization++
- Turing Kernelization
- Lossy Kernelization

Lecture #14

Roohani Sharma
February 08, 2021

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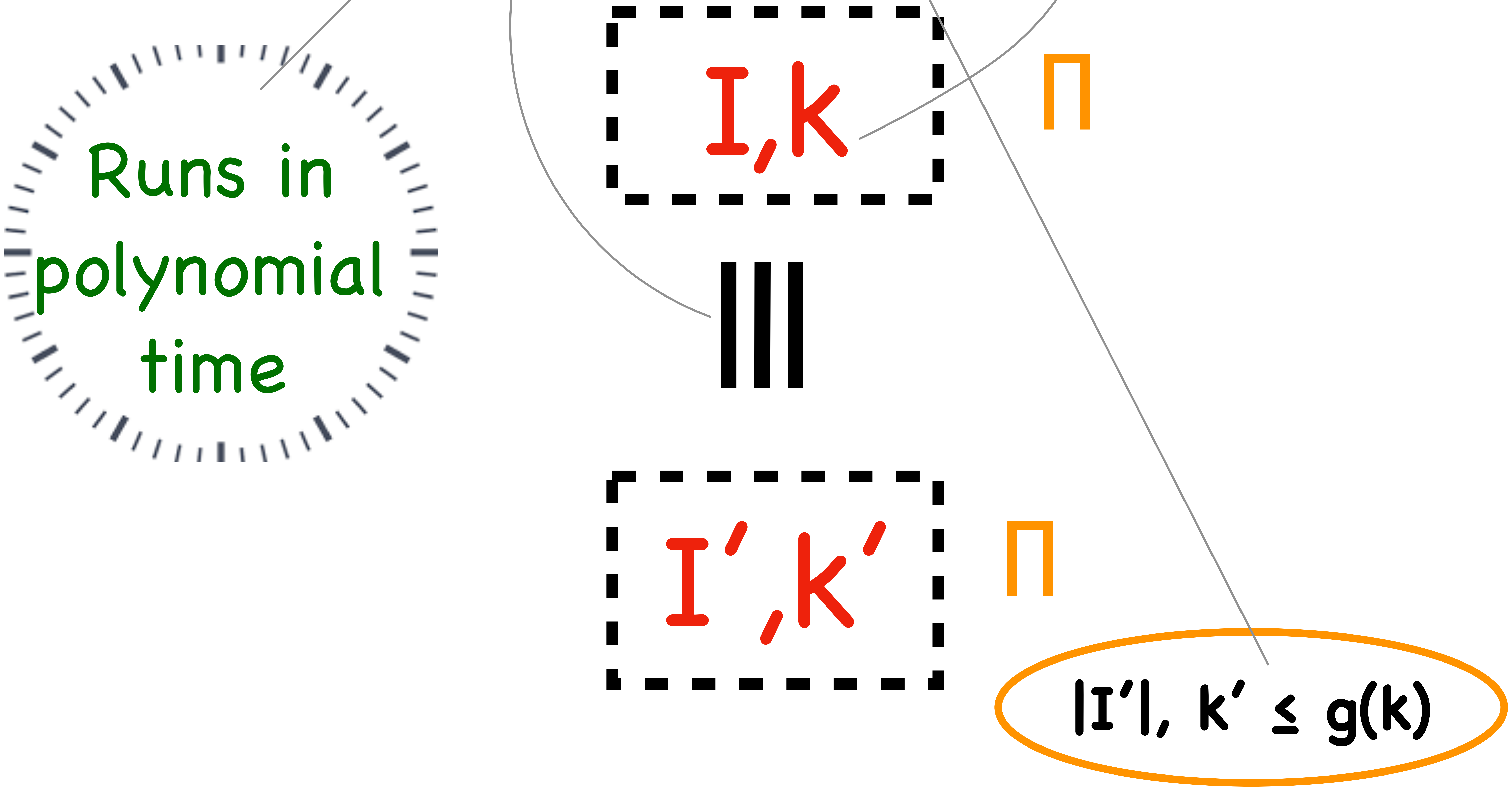
Kernelization

- Efficient pre-processing with guarantees for parameterized problems



Π admits a kernel of size g(k).

If g(k) is a polynomial/exponential function, then Π admits a polynomial/exponential kernel.

OR-composition

Let L be a parameterized problem.

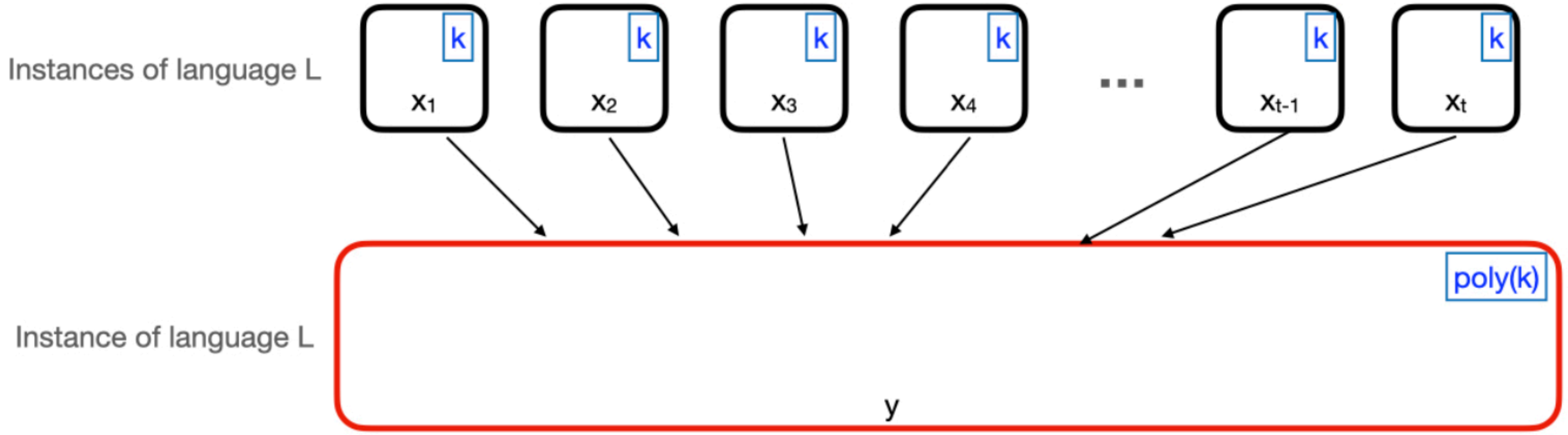
OR-composition for L

Input: $(x_1, k), \ldots, (x_t, k)$ such that $x_i \in \Sigma^*$ and k is a non-negative integer.

Output: (y, k^*) such that

- $(y, k^*) \in L$ if and only if $(x_i, k) \in L$ for some i, and
- $k^* = poly(k)$.

Time: polynomial in the input, that is $poly(\sum_{i=1}^{t} |x_i| + k)$.



Eg. k-PATH, STEINER TREE, MAX LEAF SUBGRAPH (Exercise #03) do not admit poly kernel.

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Parameter: k

Question: Does G have a subtree with at least k leaves ?

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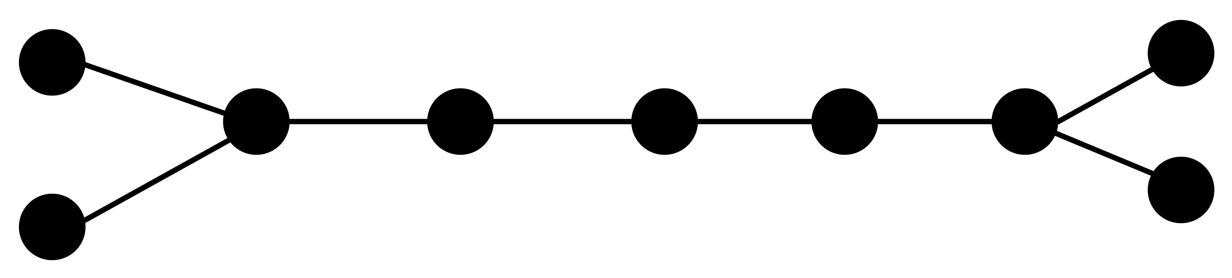

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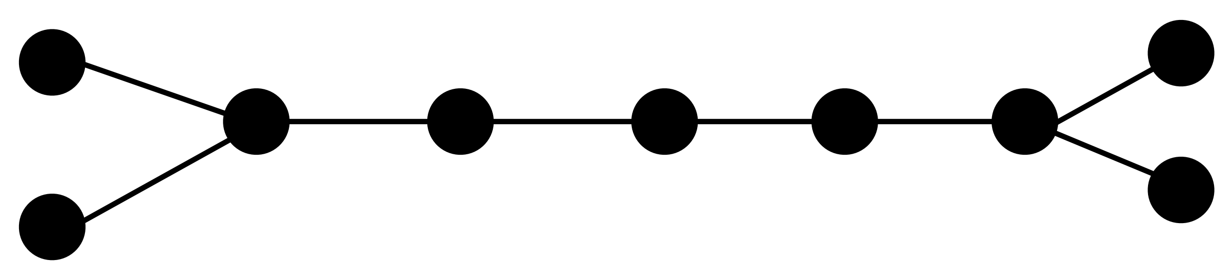

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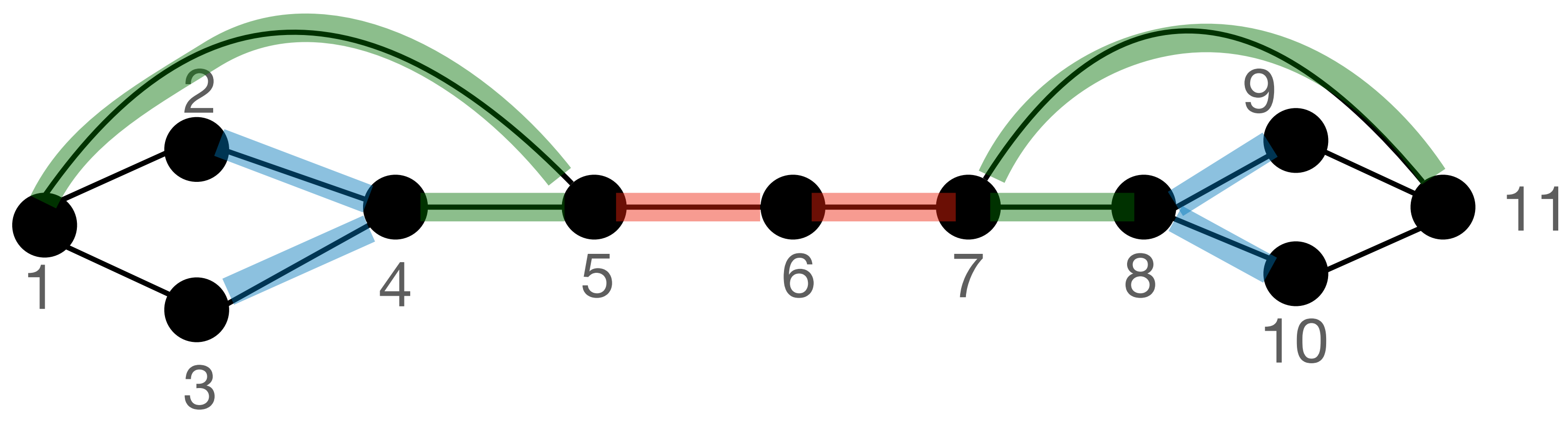

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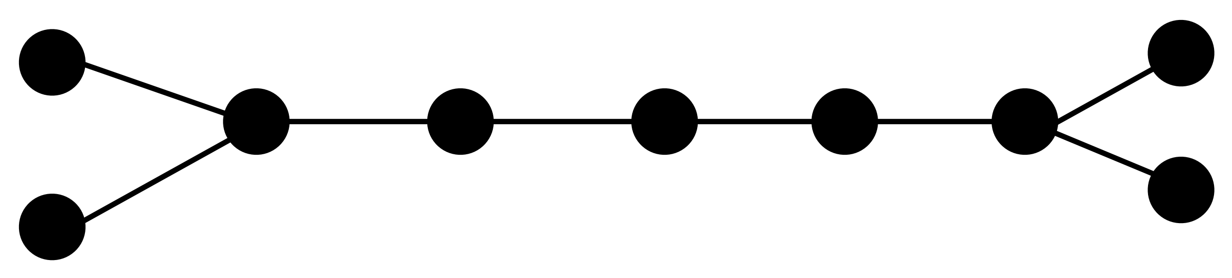

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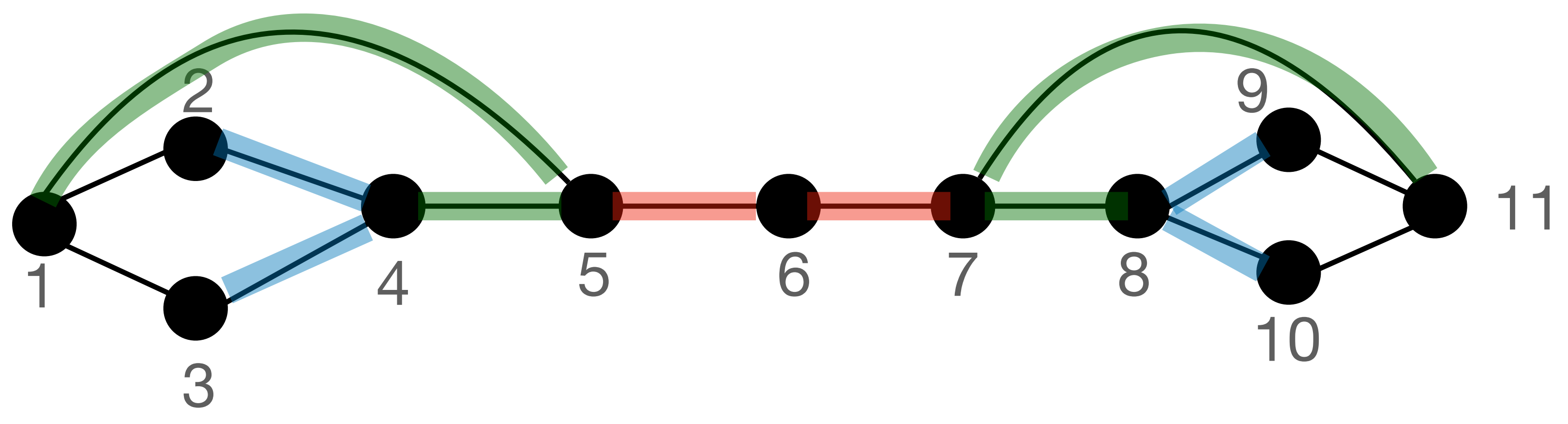

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Observe: It is not a coincidence that each solution subtree is a spanning

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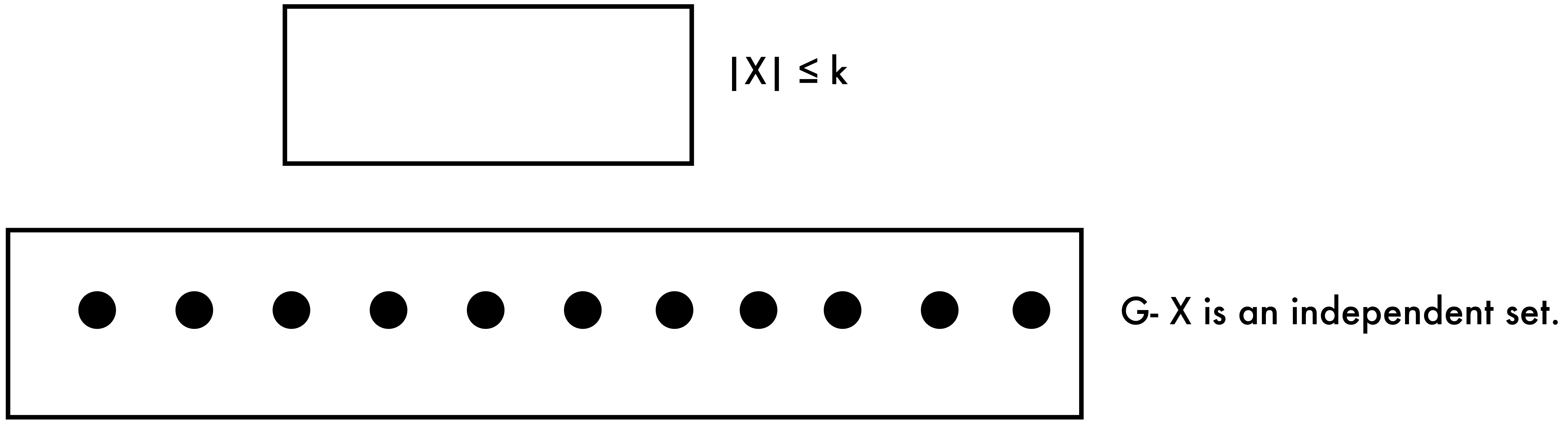

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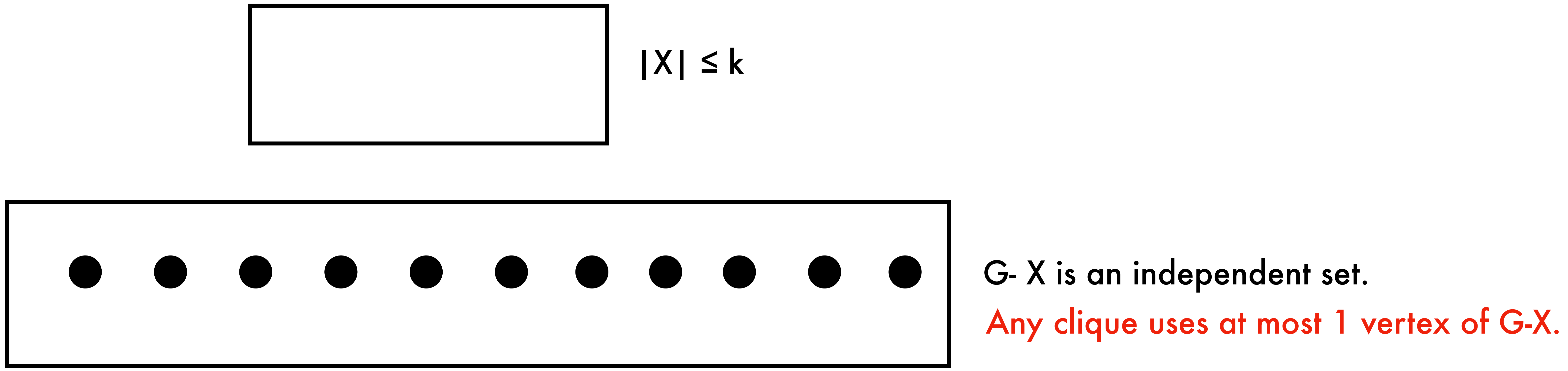

Turing Kernelization

Definition (Turing Kernel)

Let Q be a parameterized problem, and let $f : \mathbb{N} \to \mathbb{N}$ be a computable function. A **Turing Kernel** for Q of size f is an algorithm that can decide if an instance of the problem is a YES instance in polynomial time, given access to an Oracle that solves instance of size $f(k)$ in unit time.

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- For CLIQUE/VC, we produced O(n) instances, each of size k+1, such that each of them can be solved independently so give an output of the input instance.
- Generally speaking, one can produce instances such that the i-th instance depends on the Oracle's answer to the previous (i-1) instances. Such kind of Turing kernels are known for k-PATH on certain graph classes.

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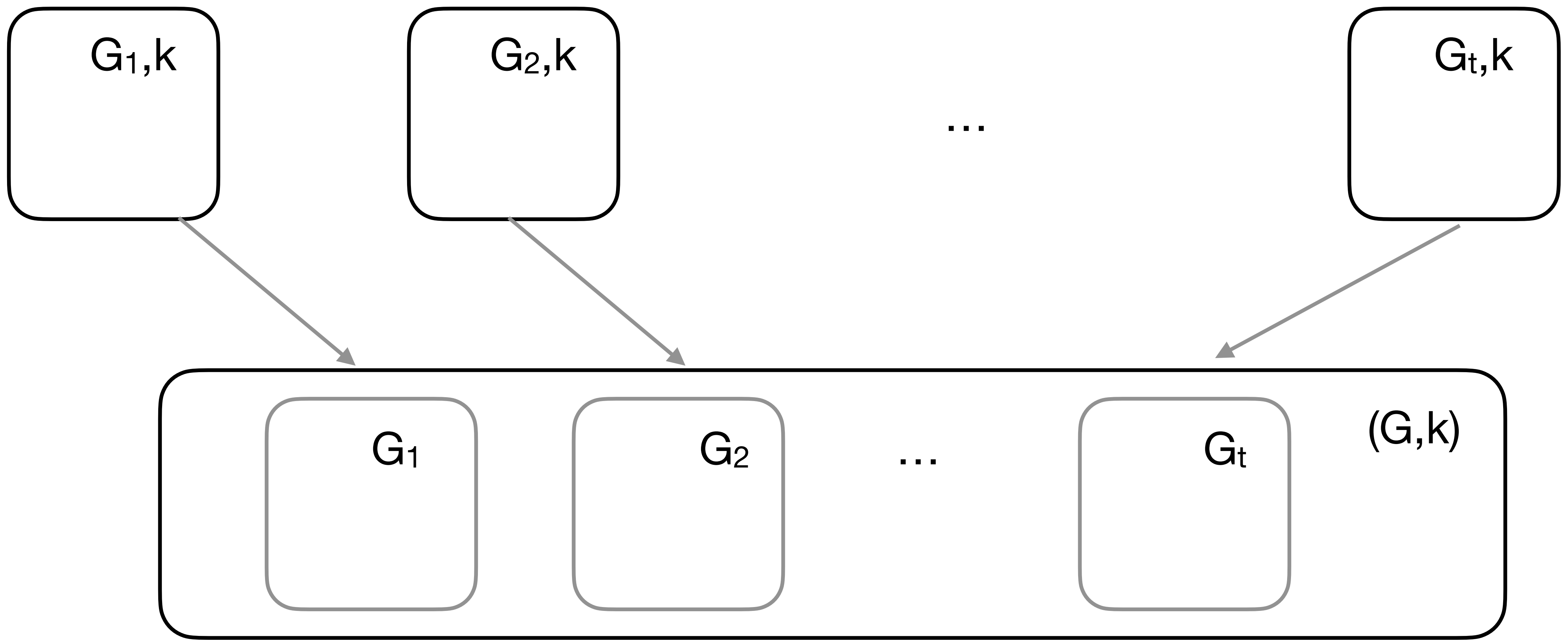

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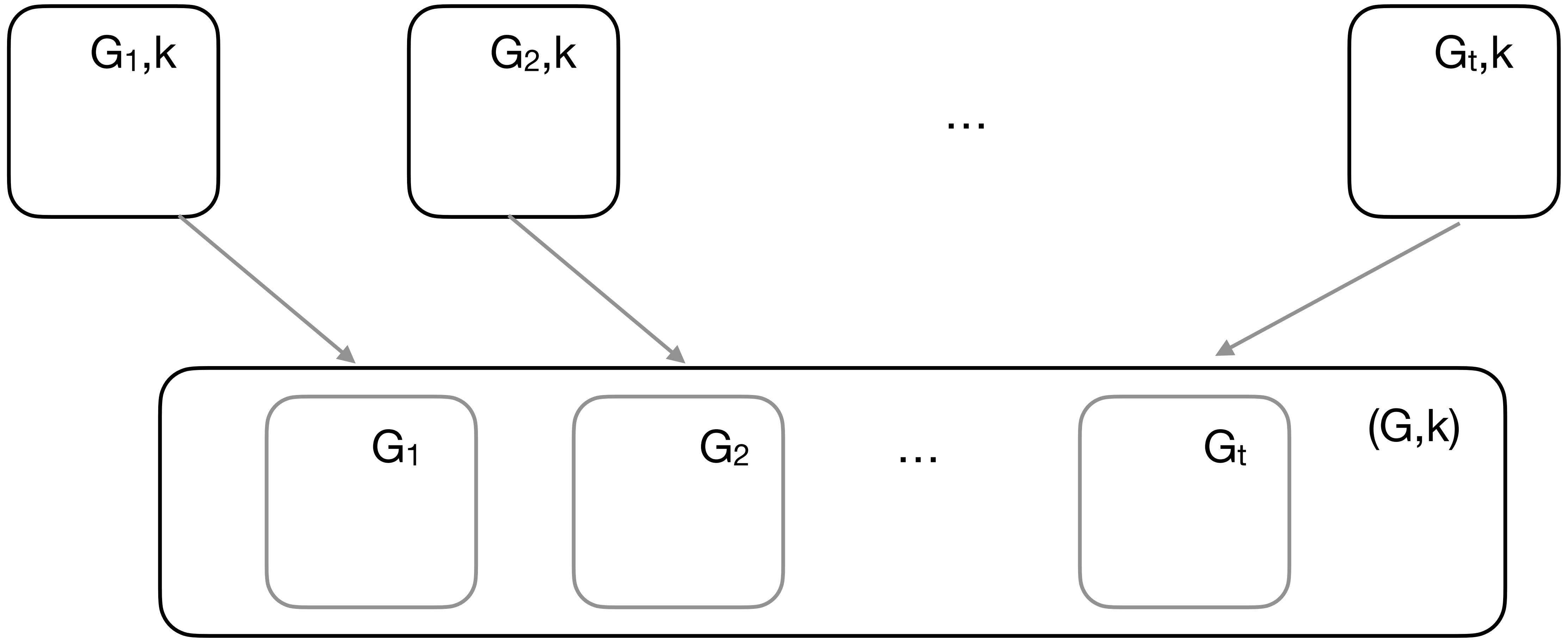

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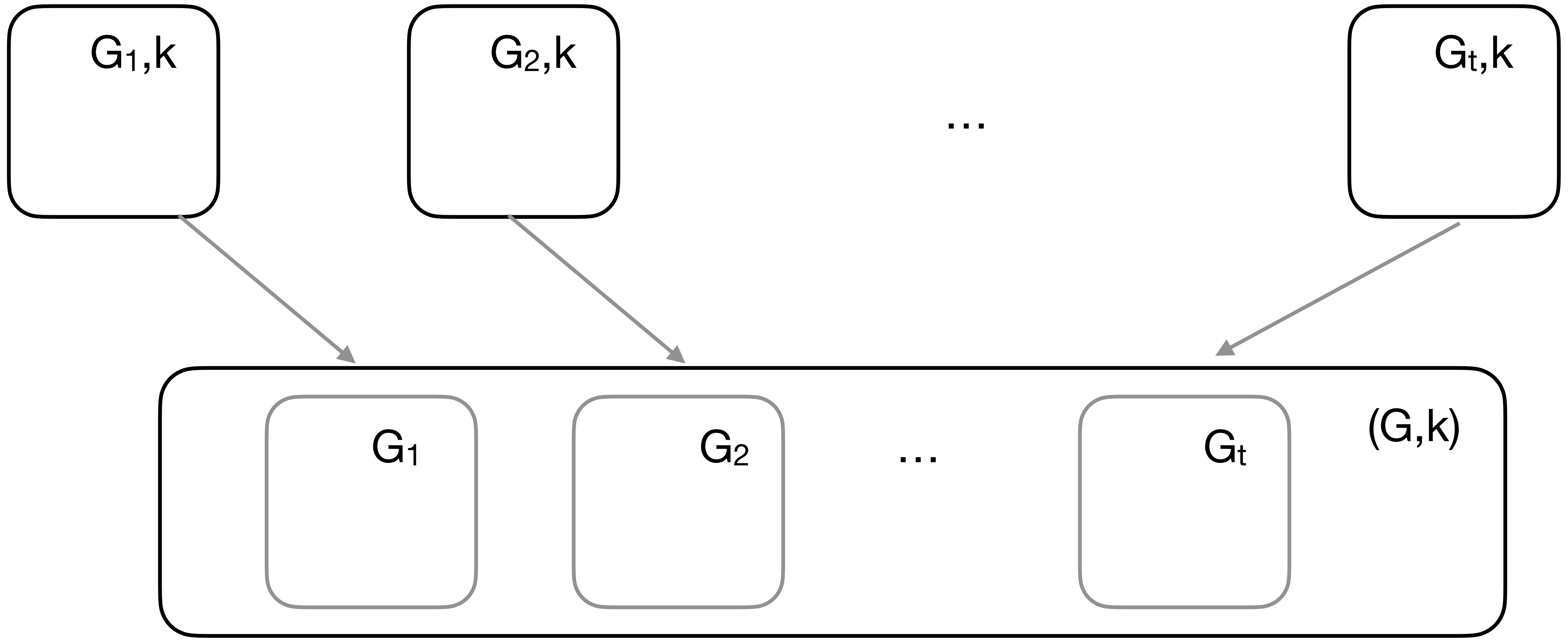

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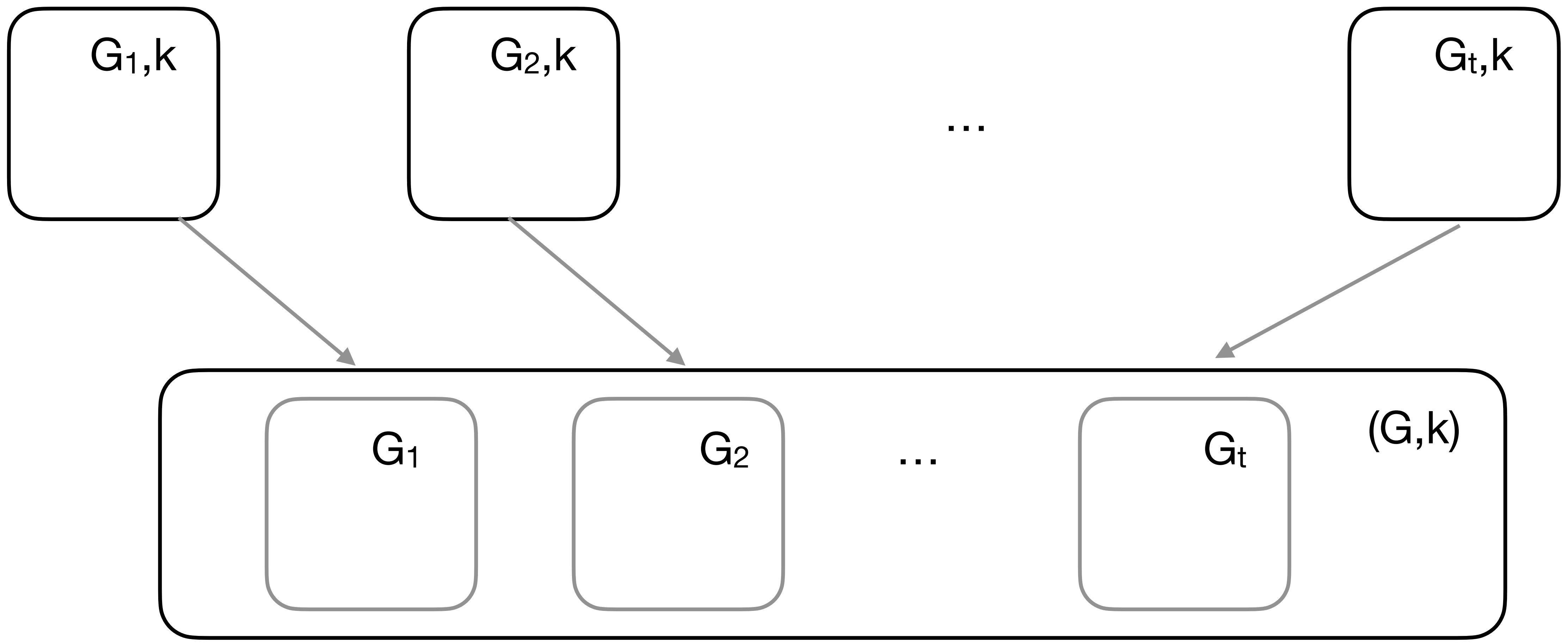

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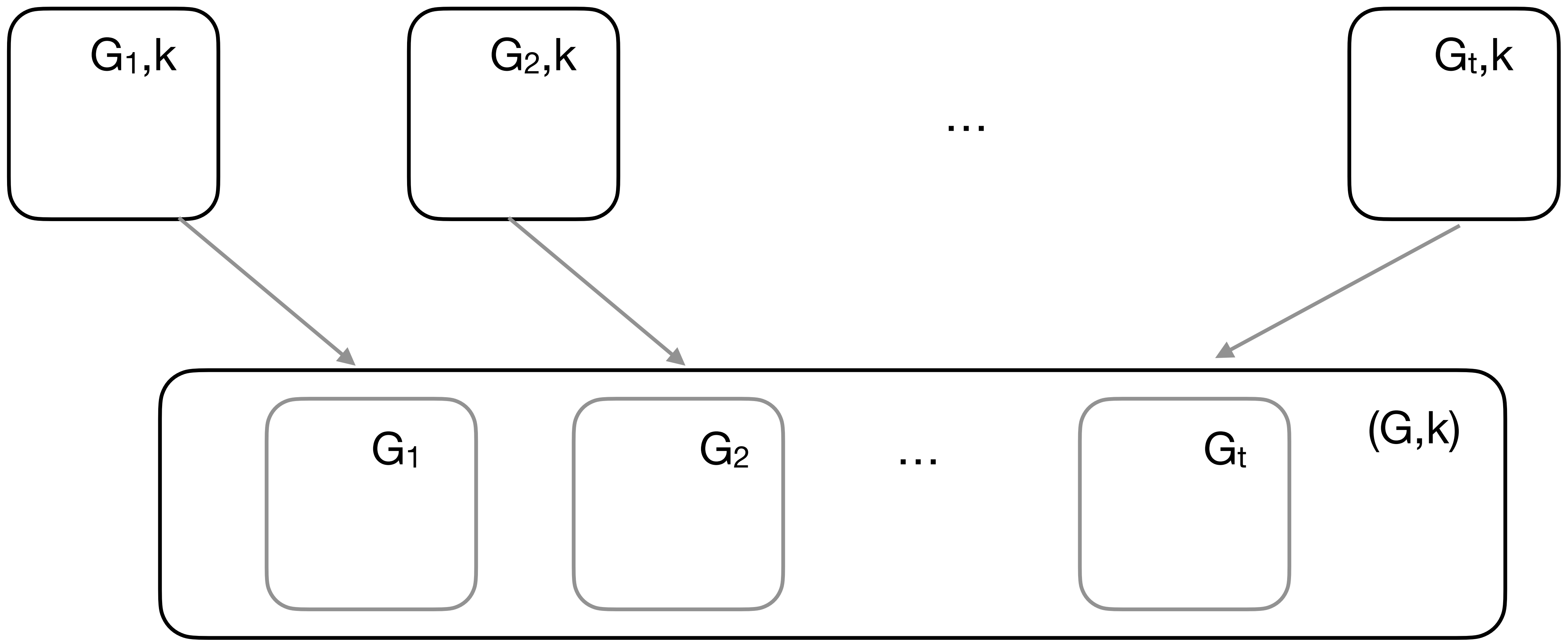

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MLS admits a polynomial Turing kernel!

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Basic reduction rules:

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Reduction rule for long degree-2 paths:

If there exists a path v_1-v_2-v_3 such that degree of each v_i is exactly 2 in G, then contract the edge v1-v2.

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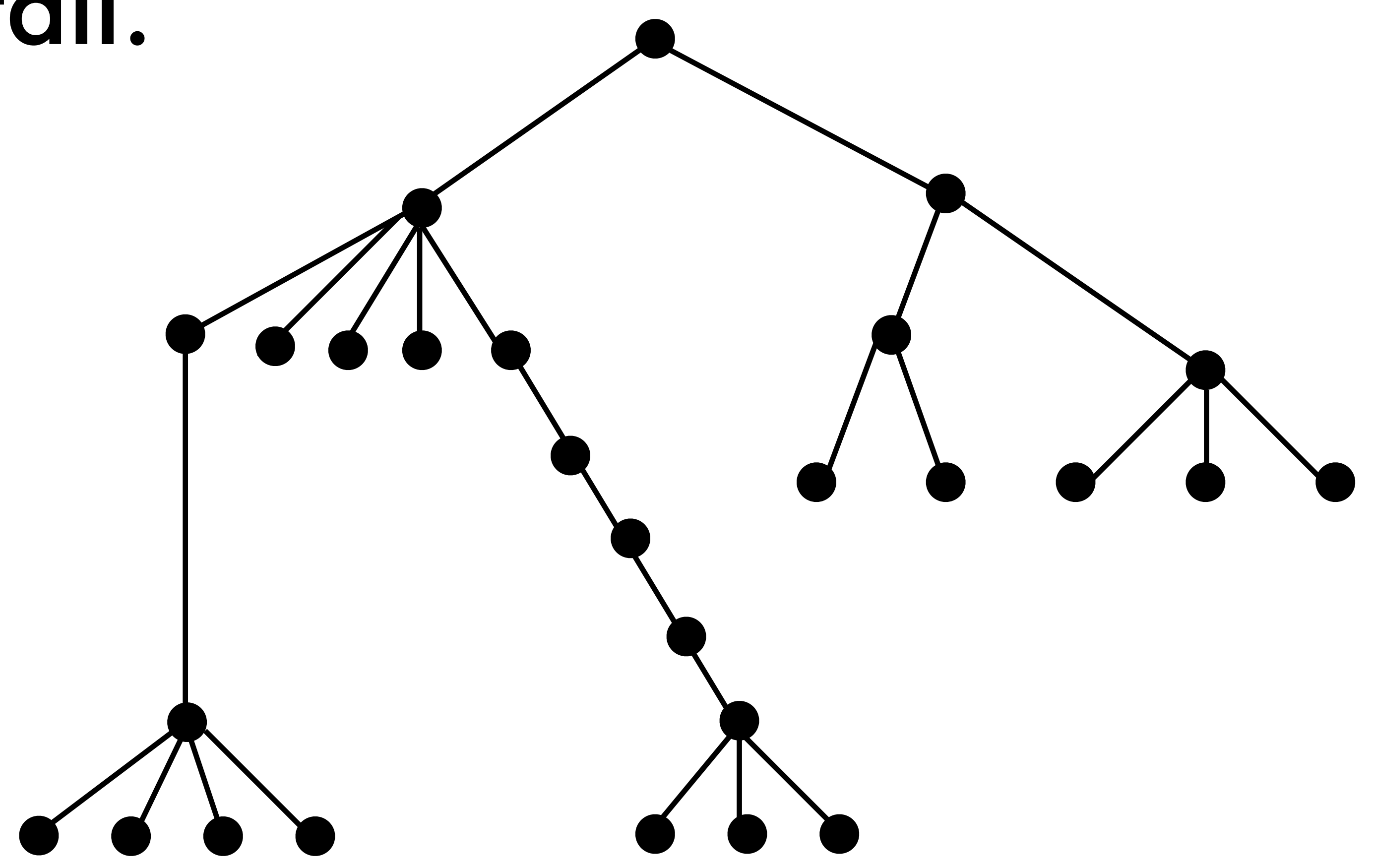

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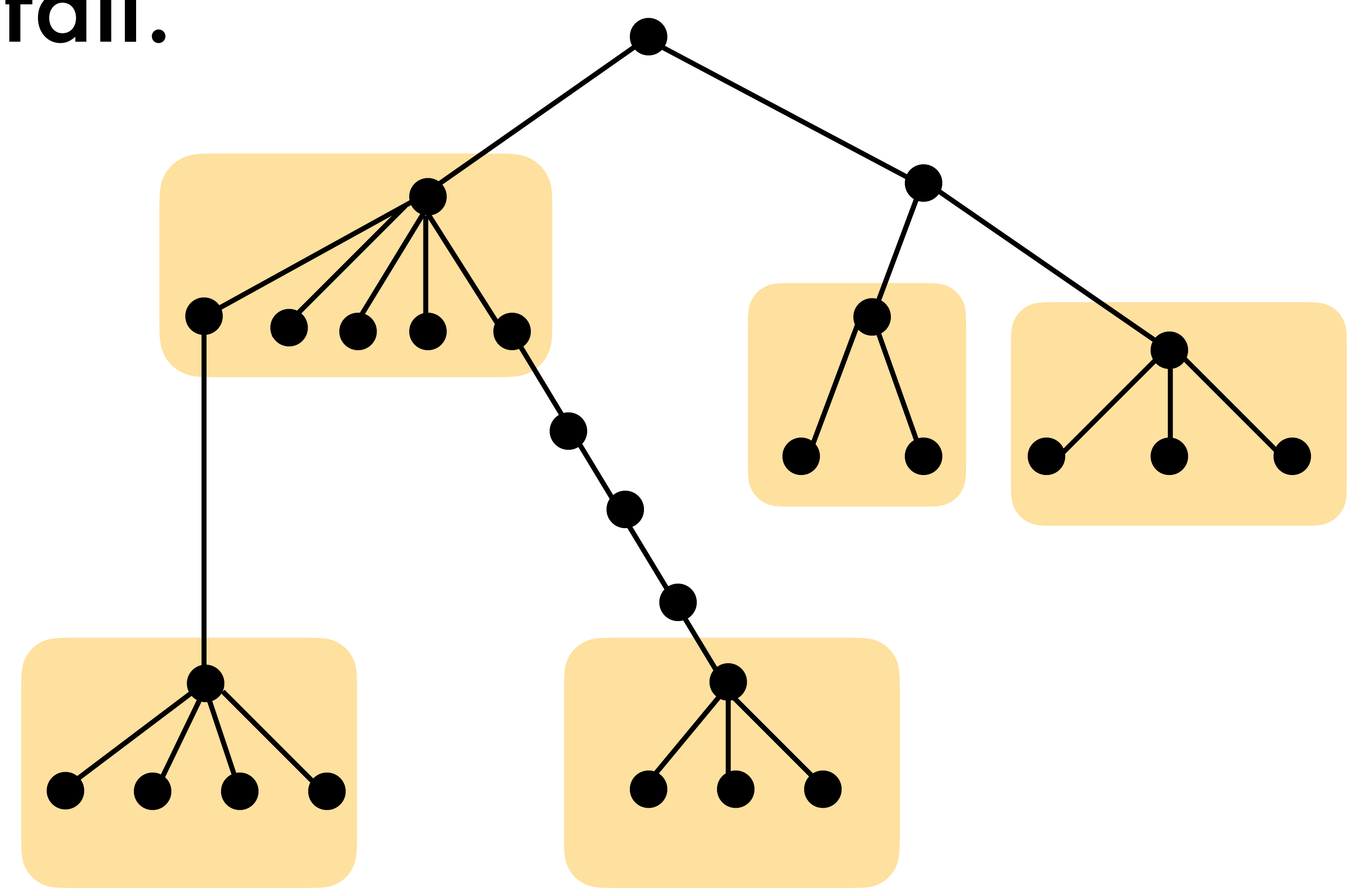

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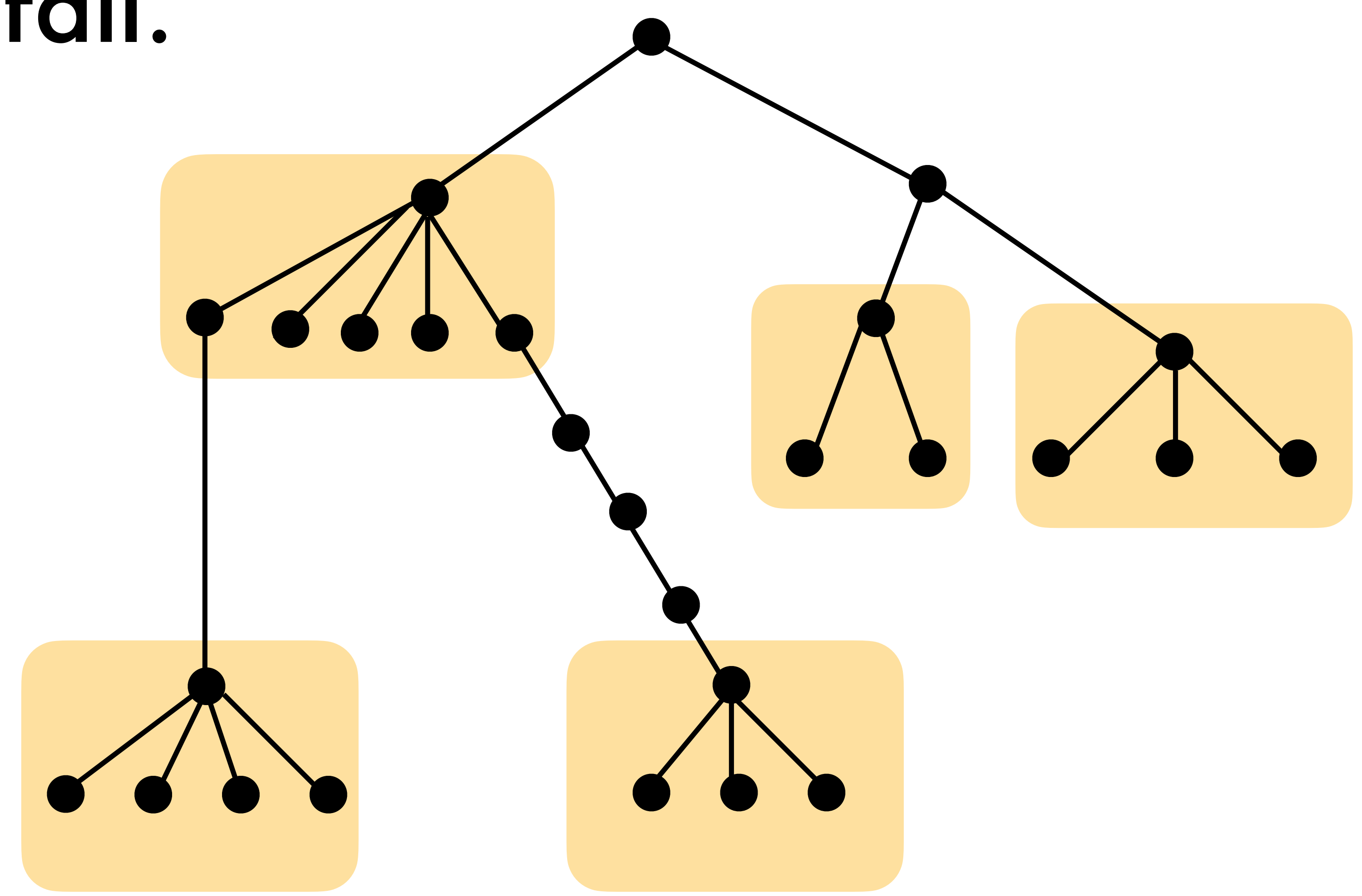

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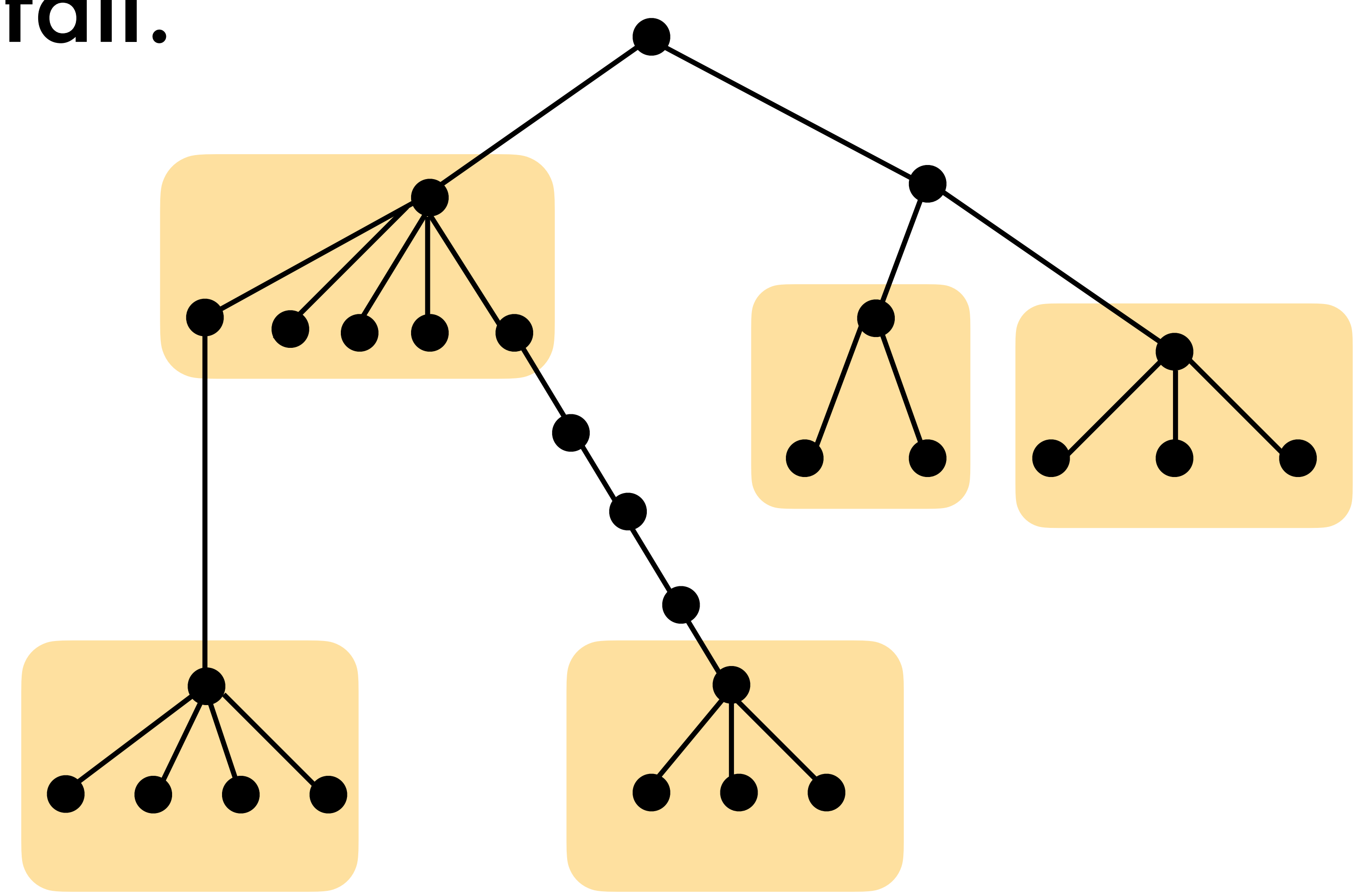

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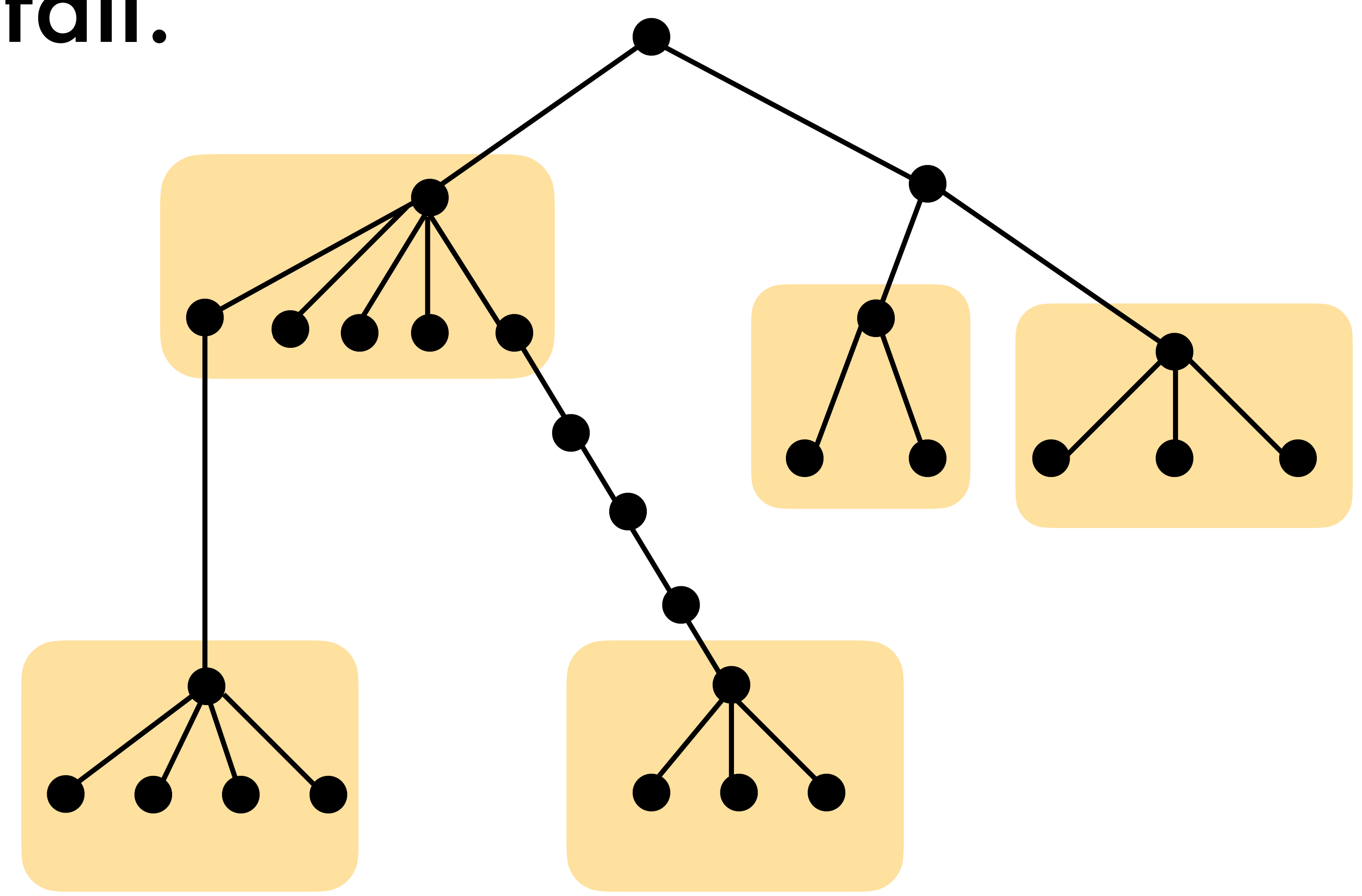

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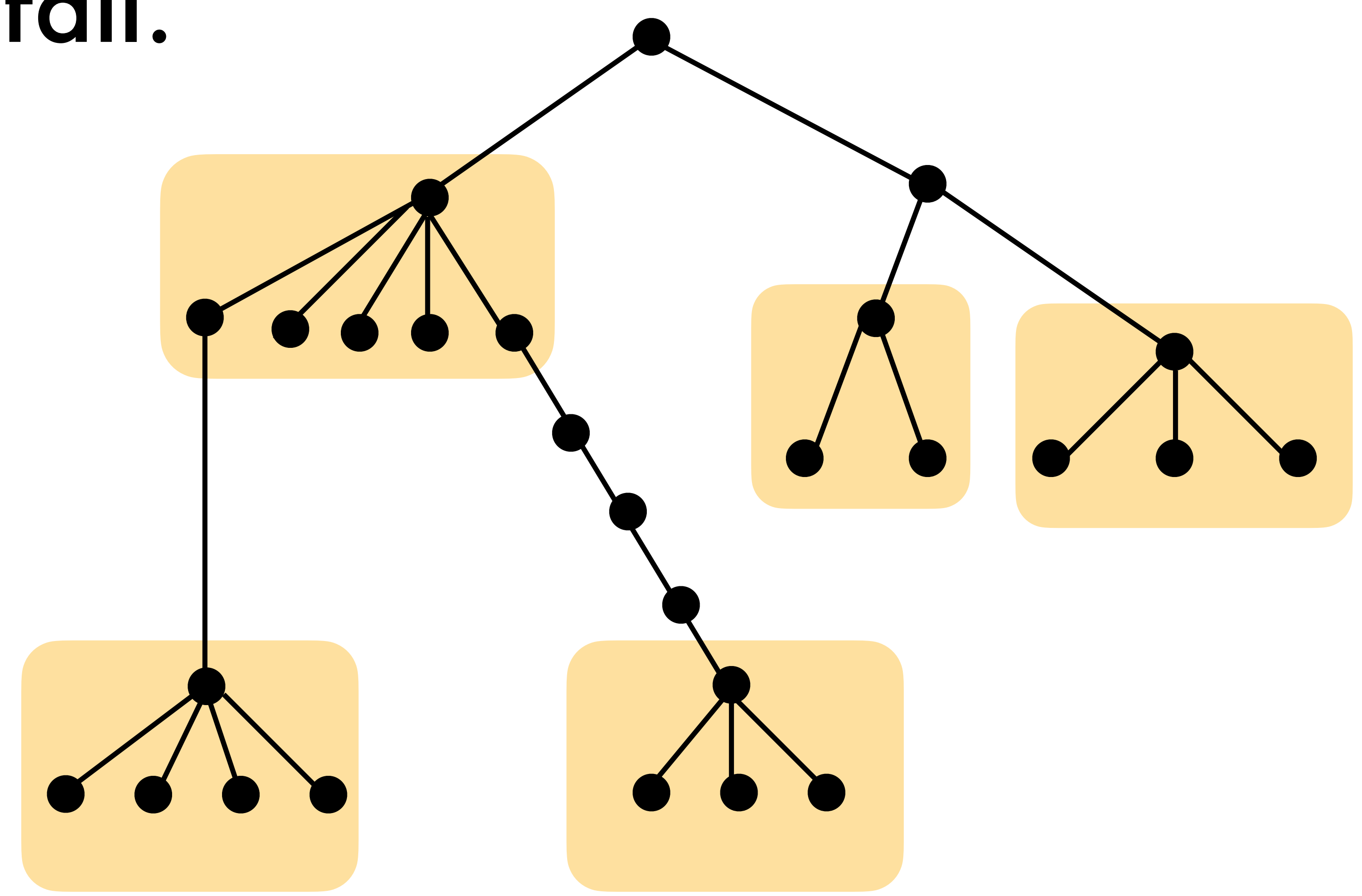

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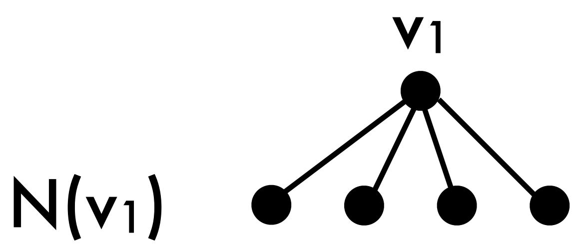

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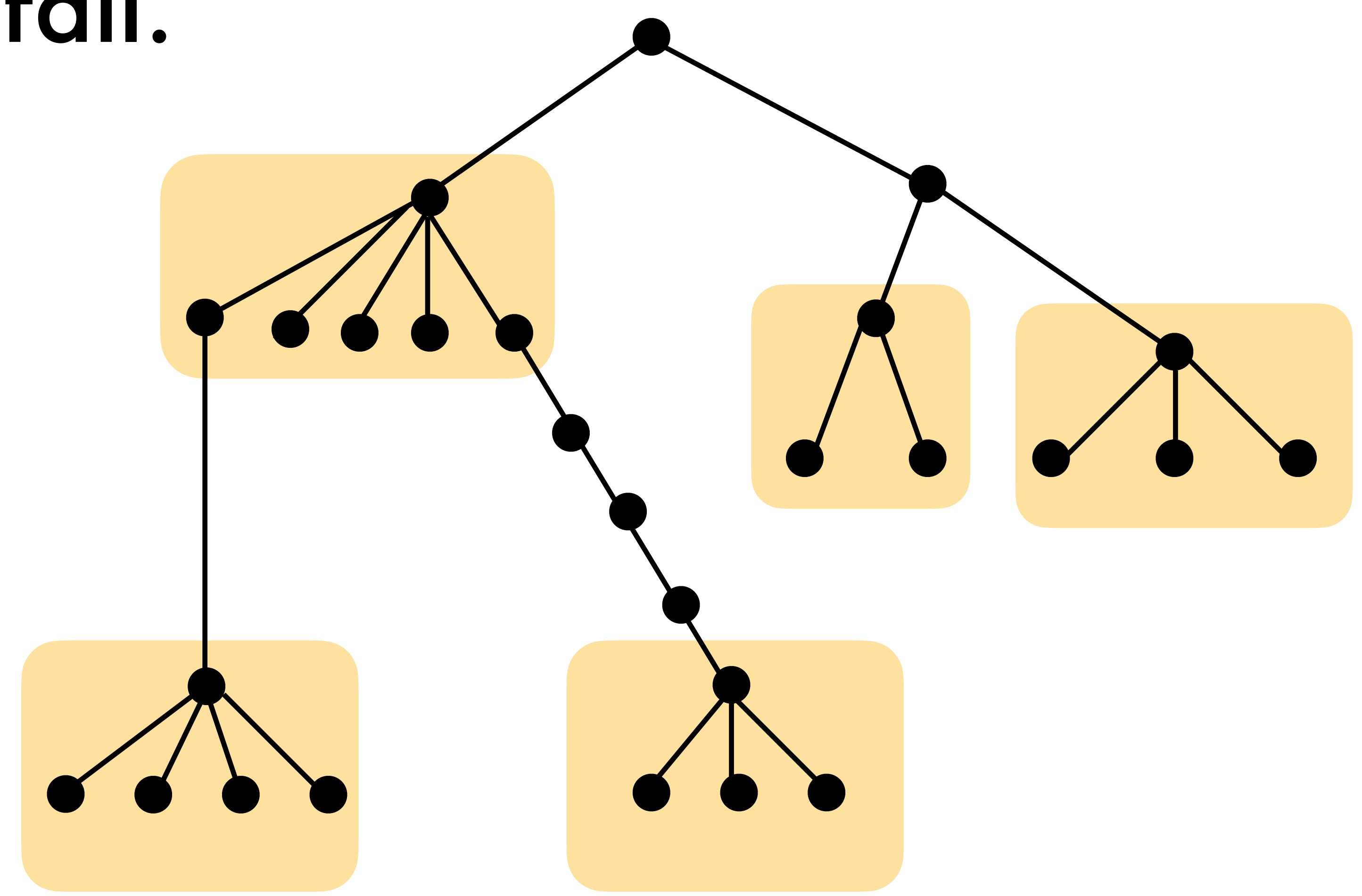

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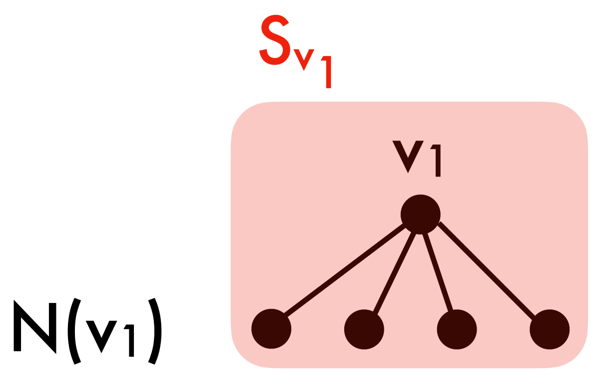

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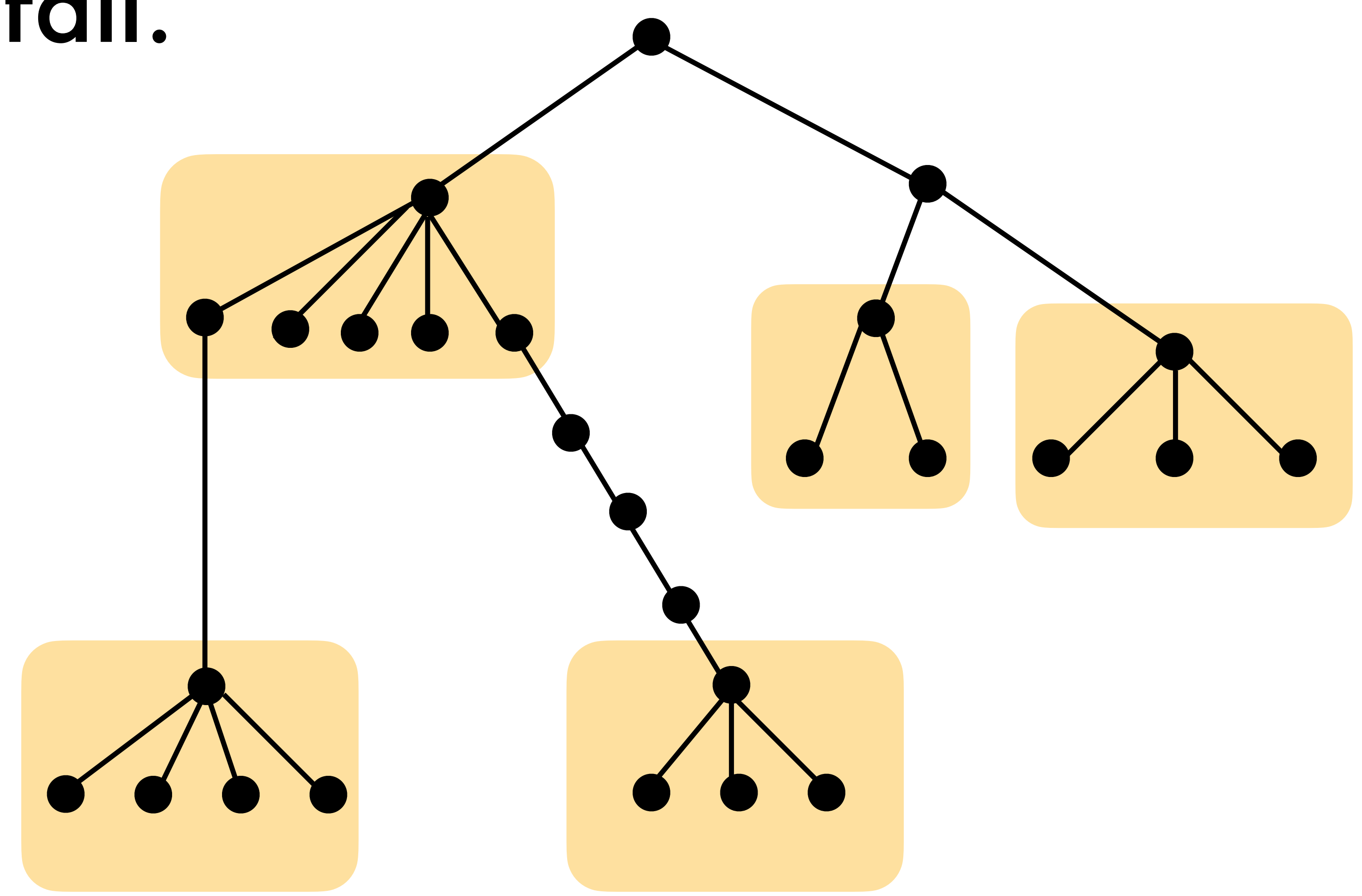

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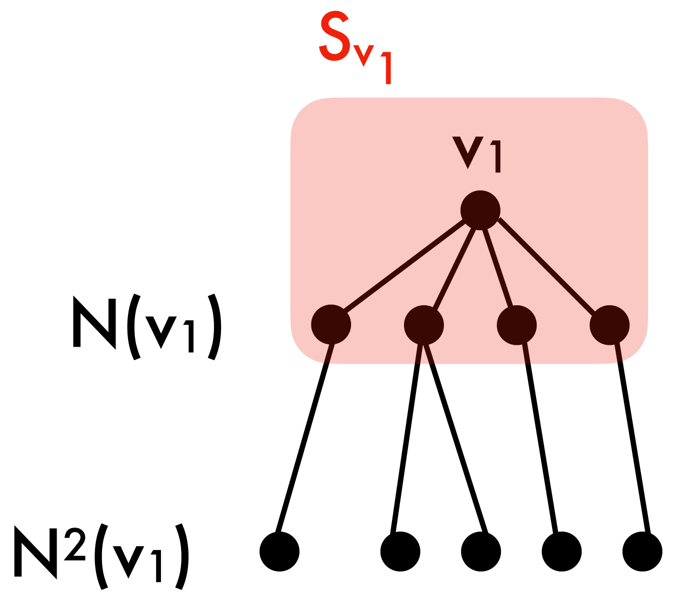

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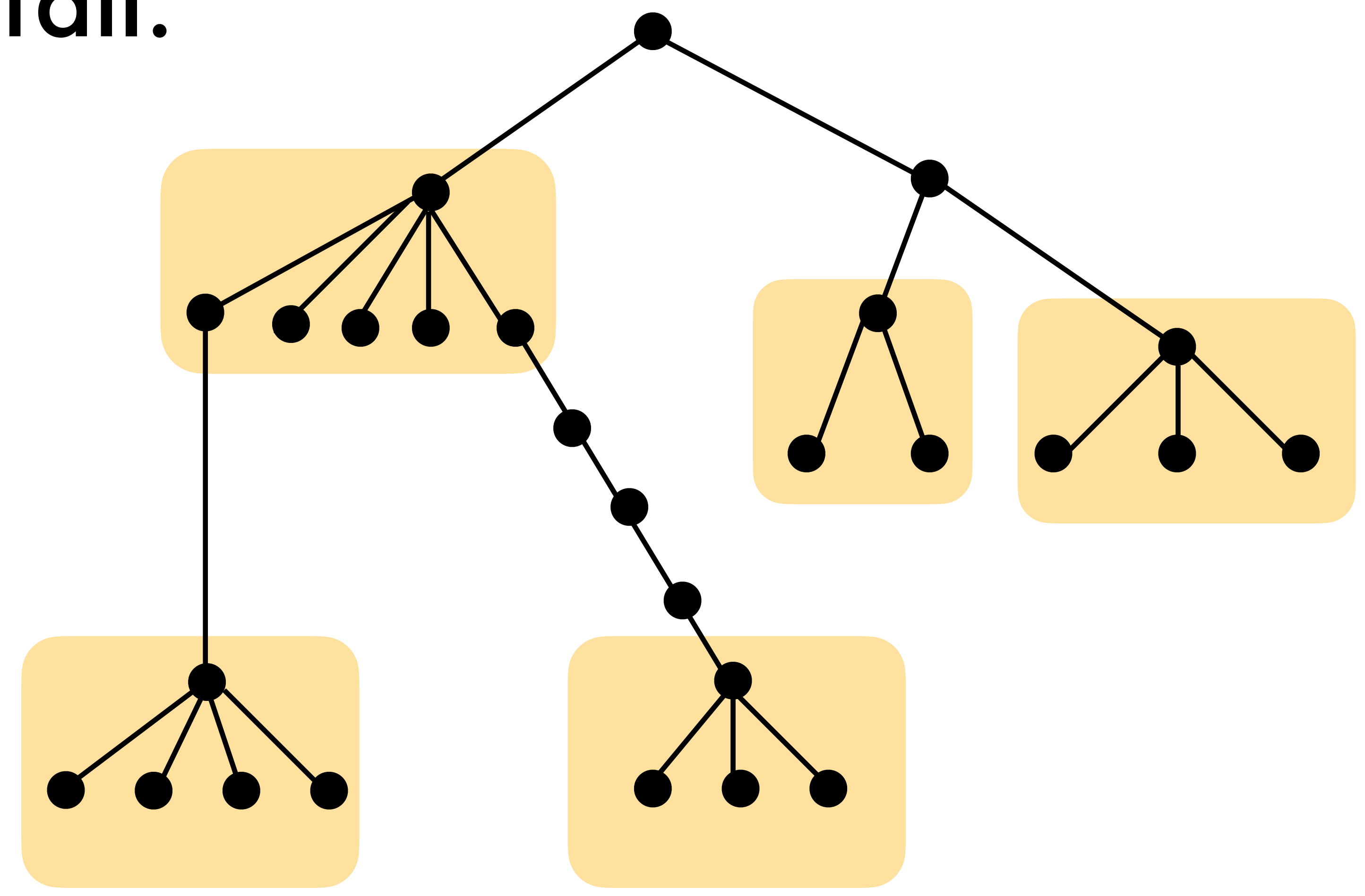

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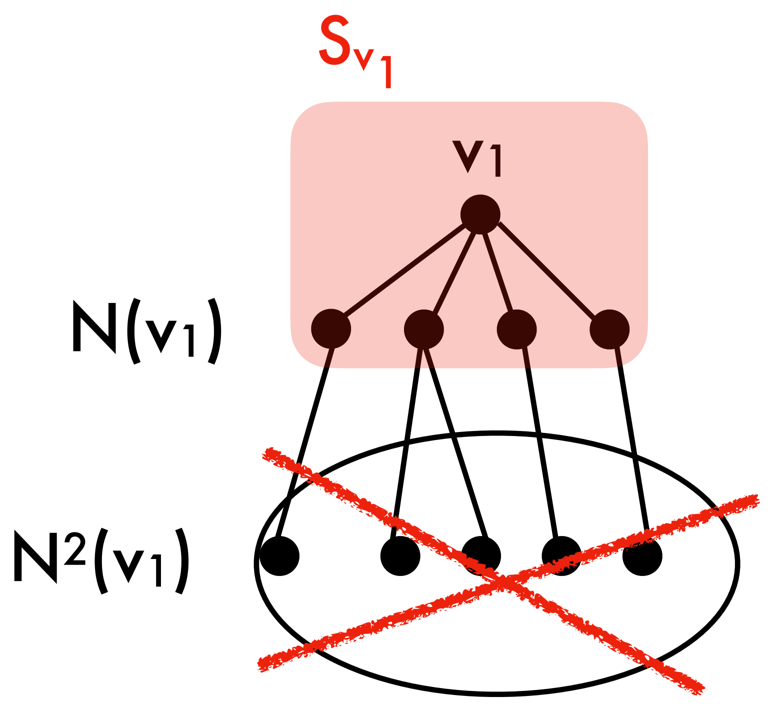

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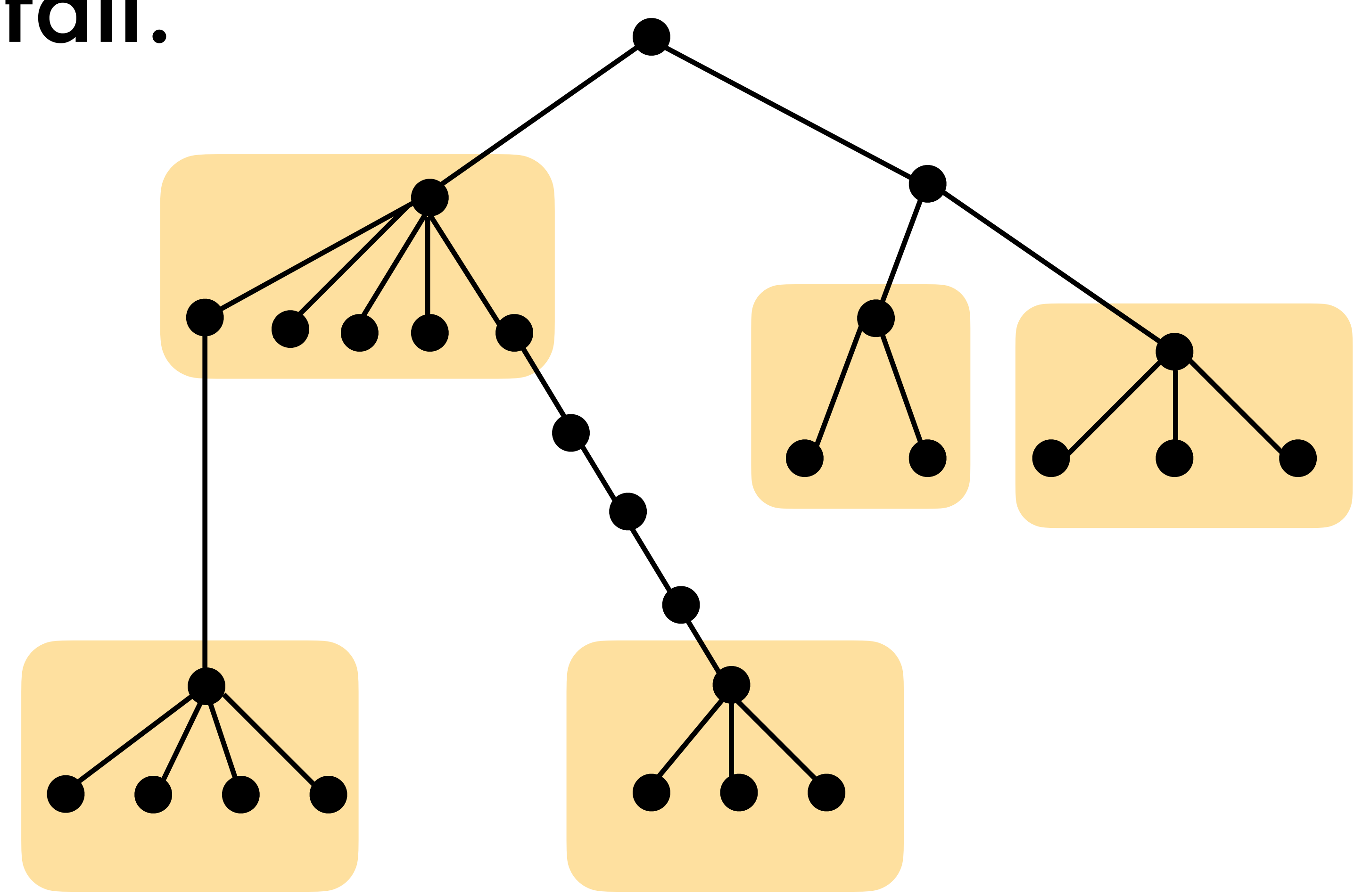

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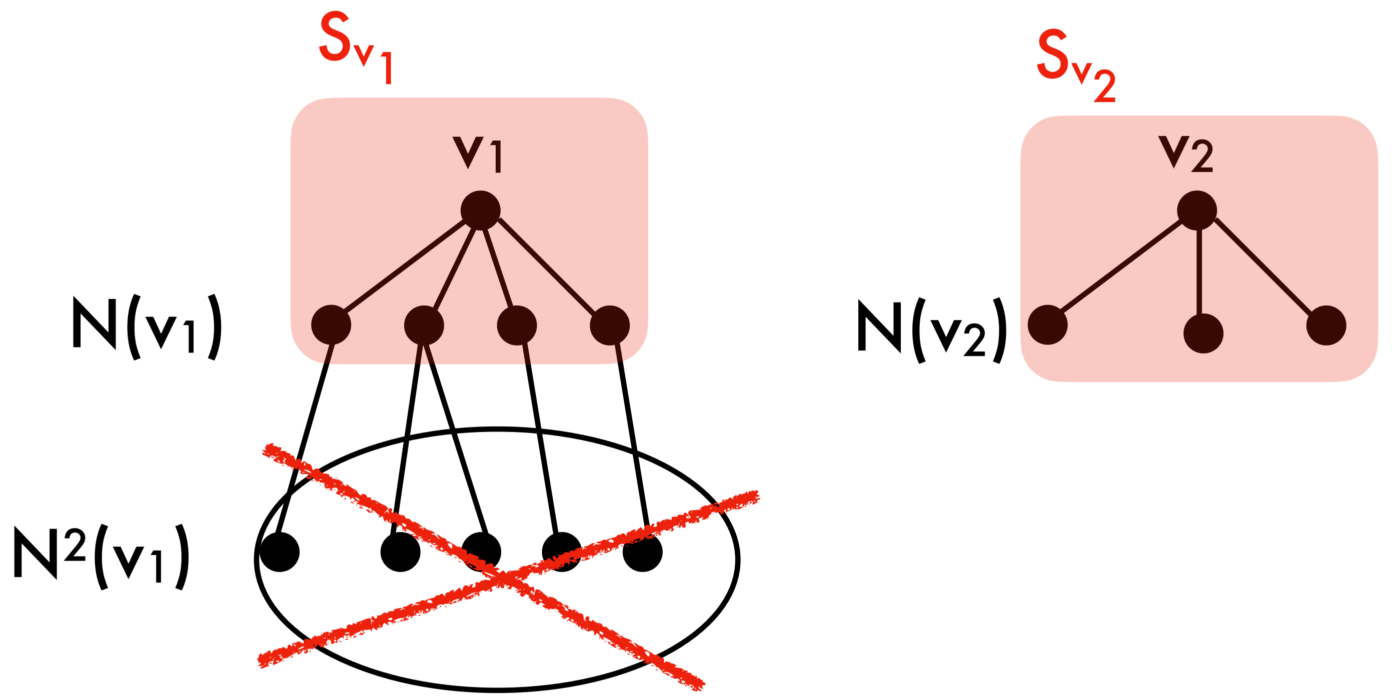

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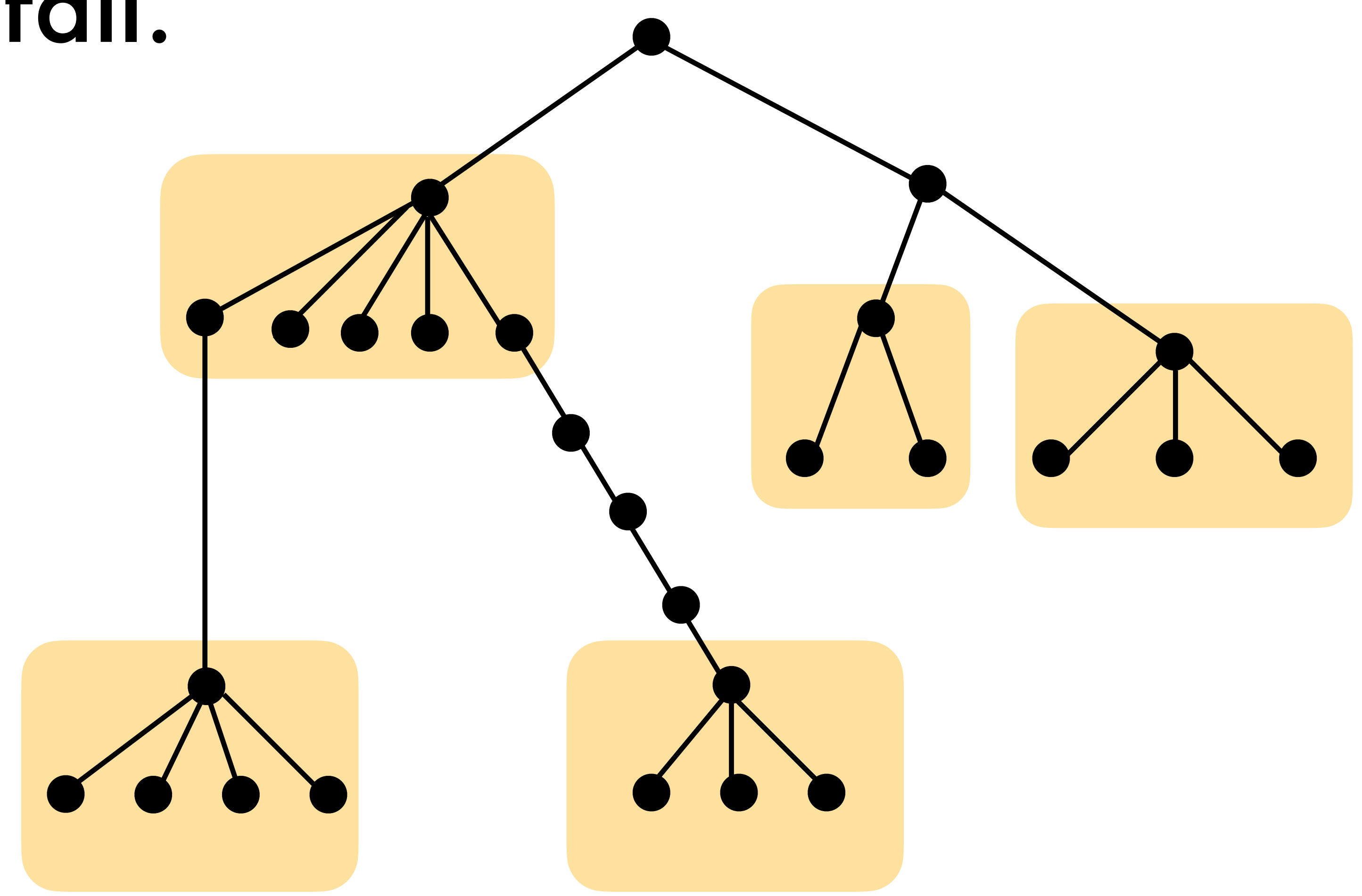

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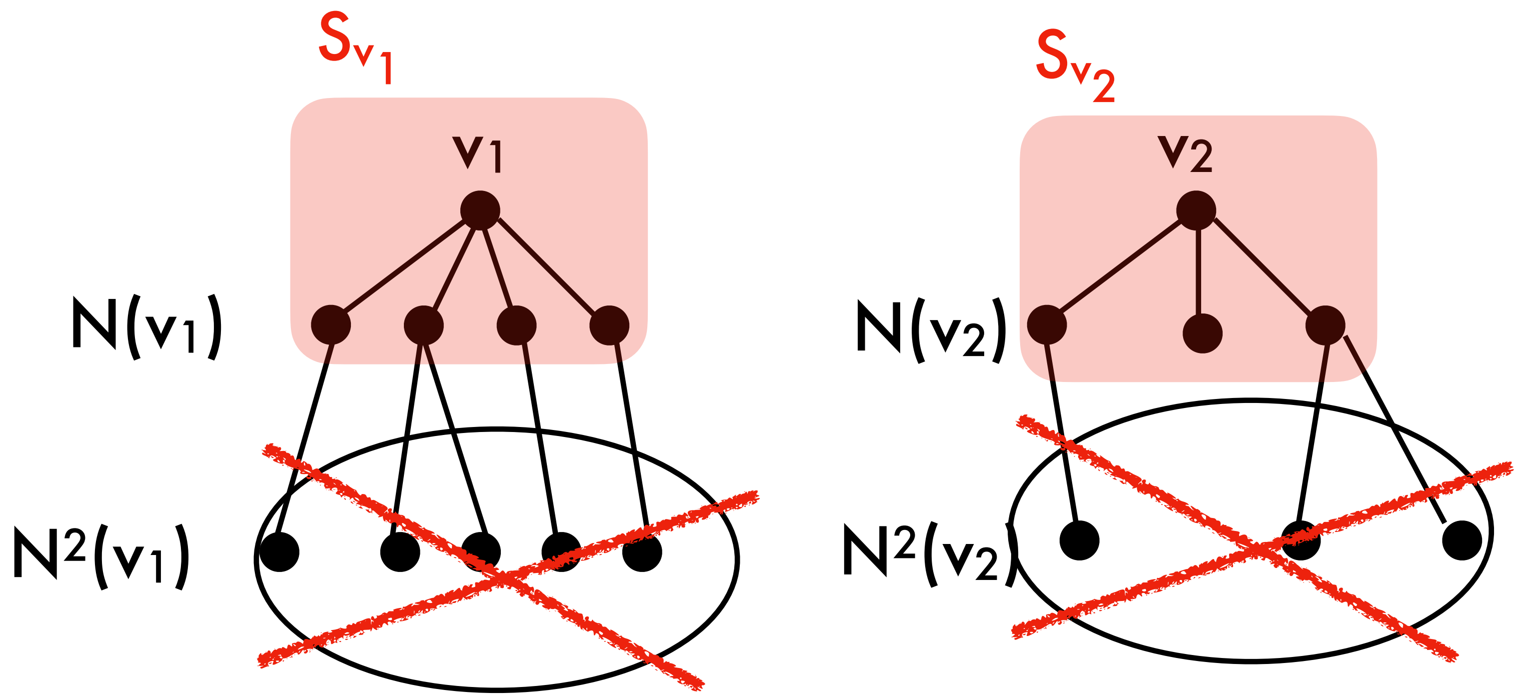

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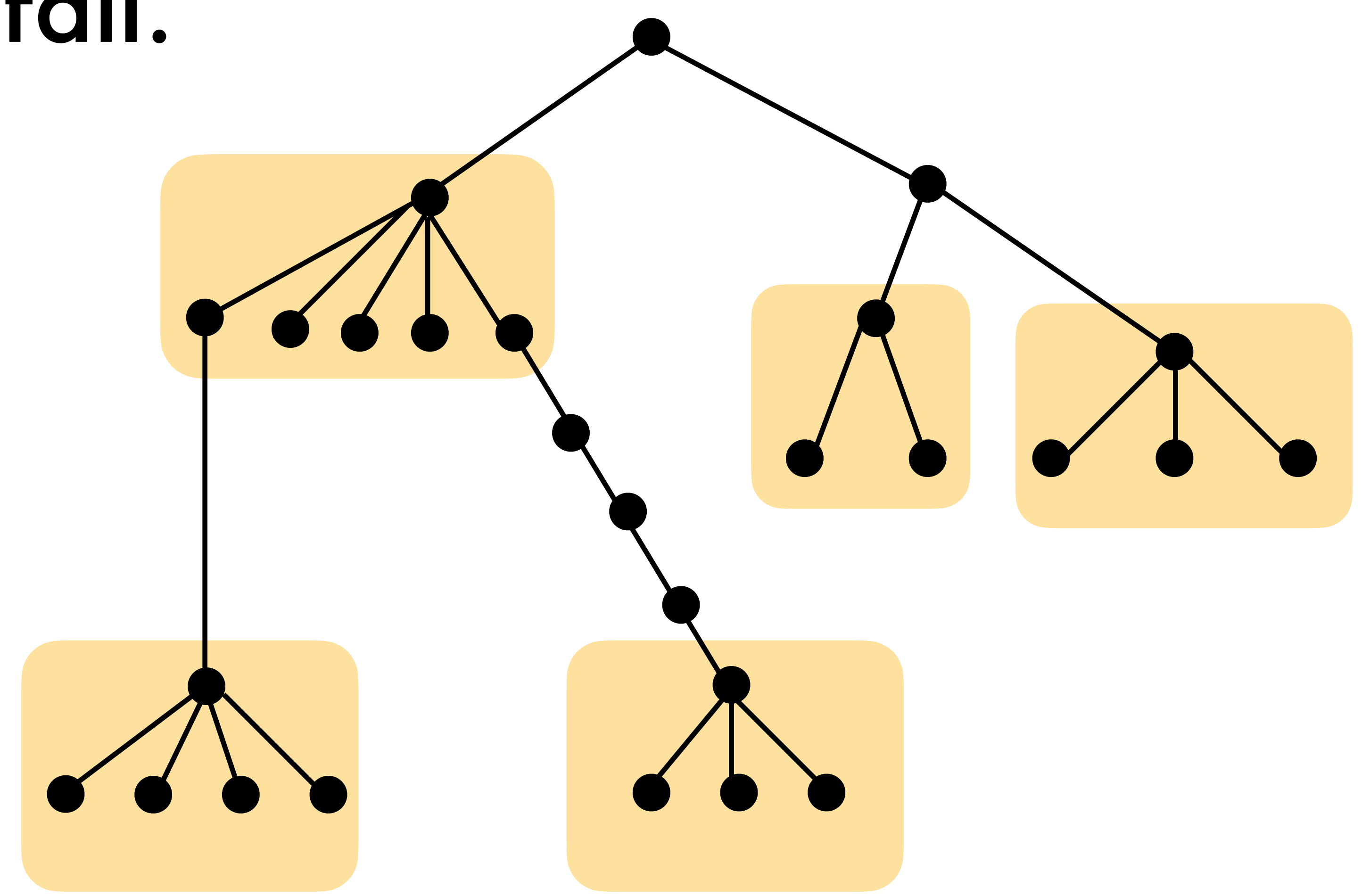

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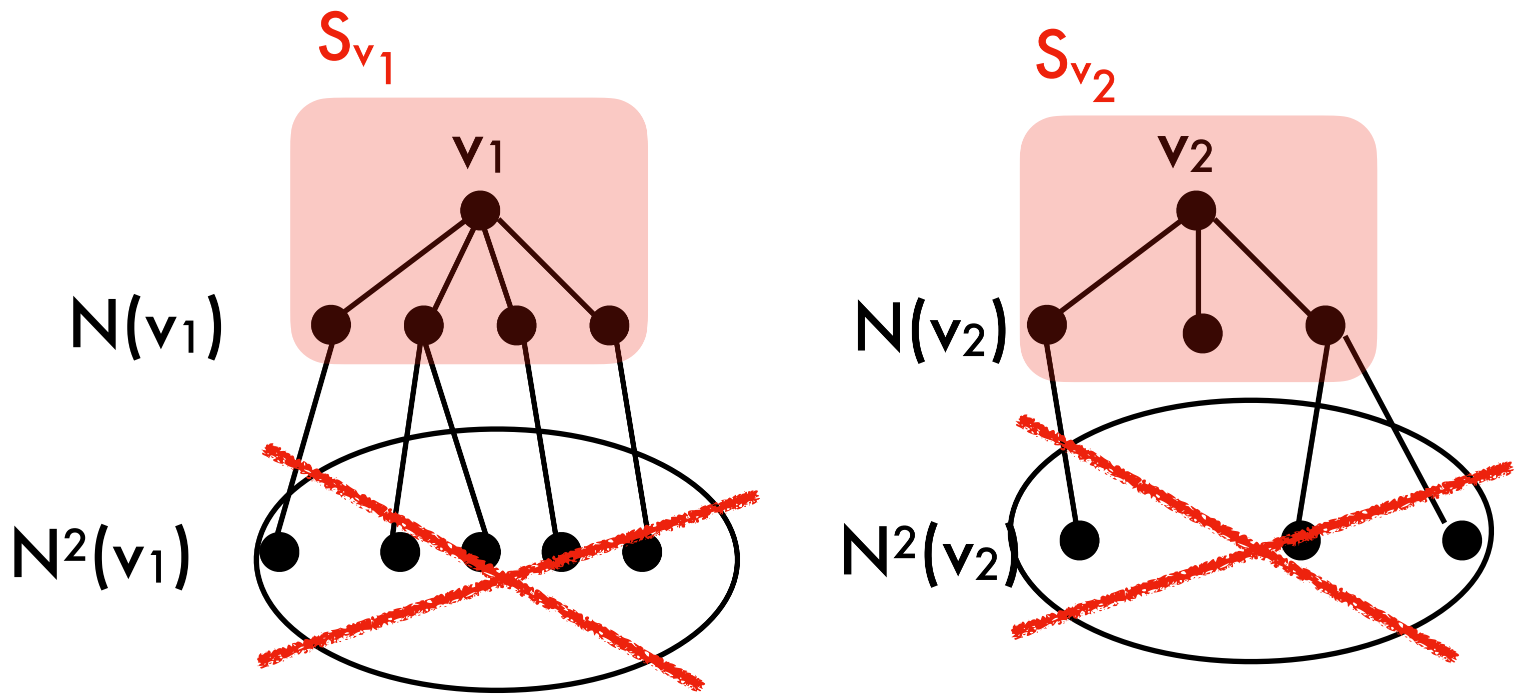

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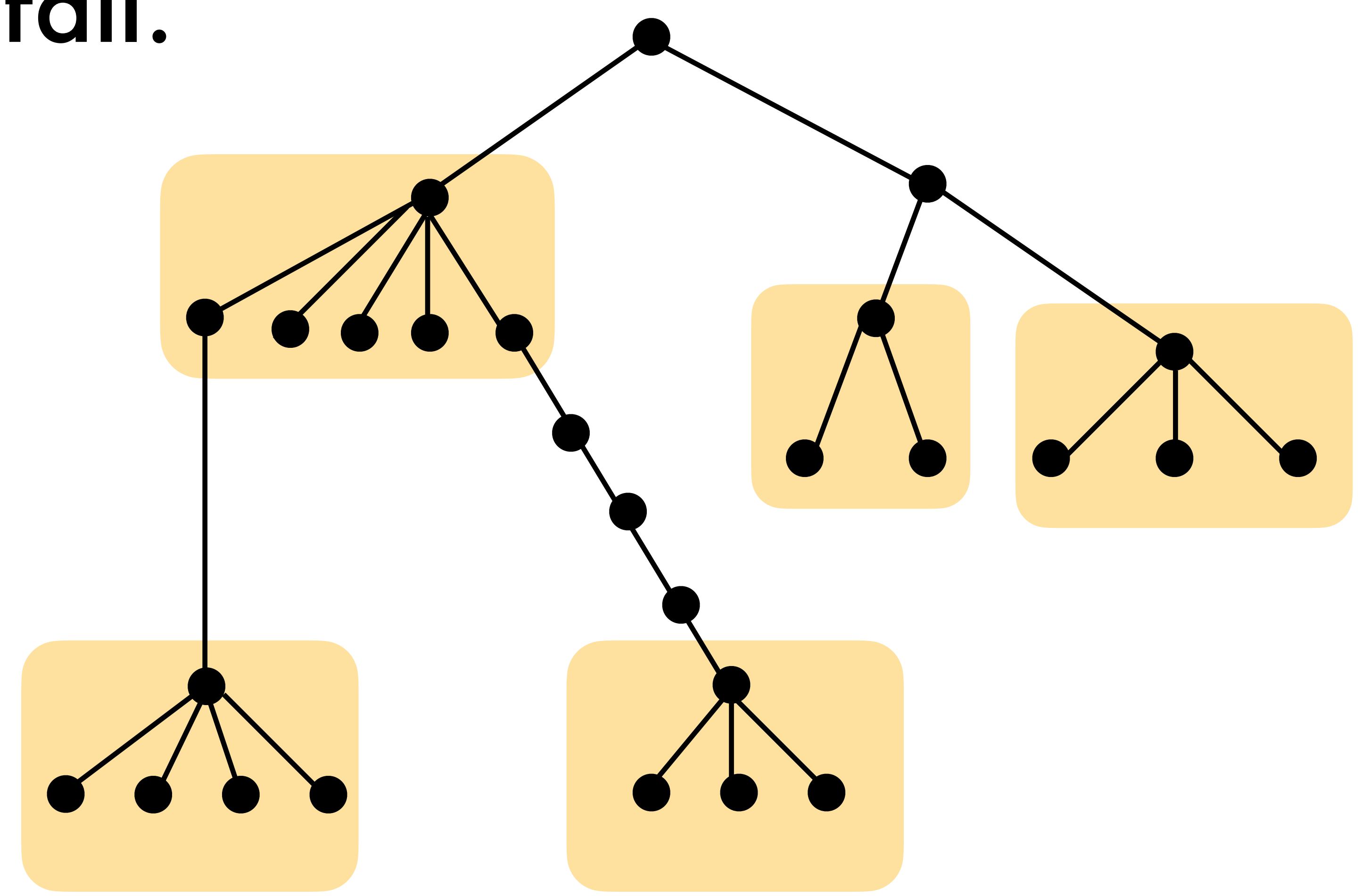

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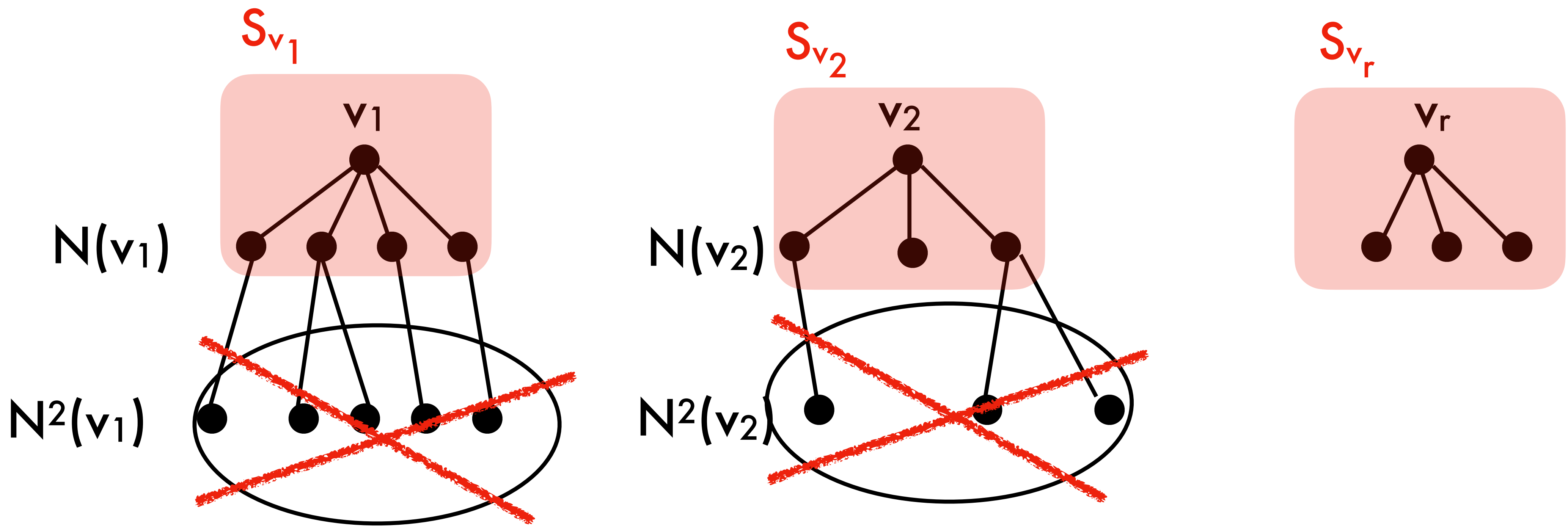
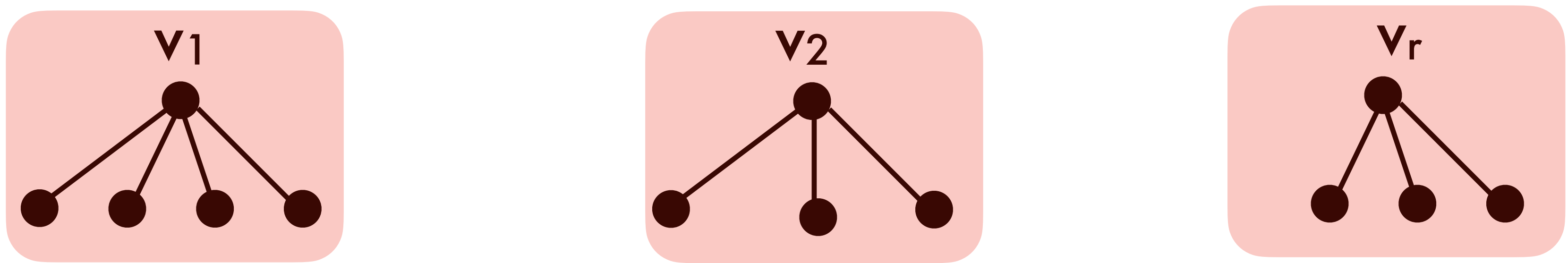


$r \geq k$

Claim: The stars in the red boxes are disjoint.

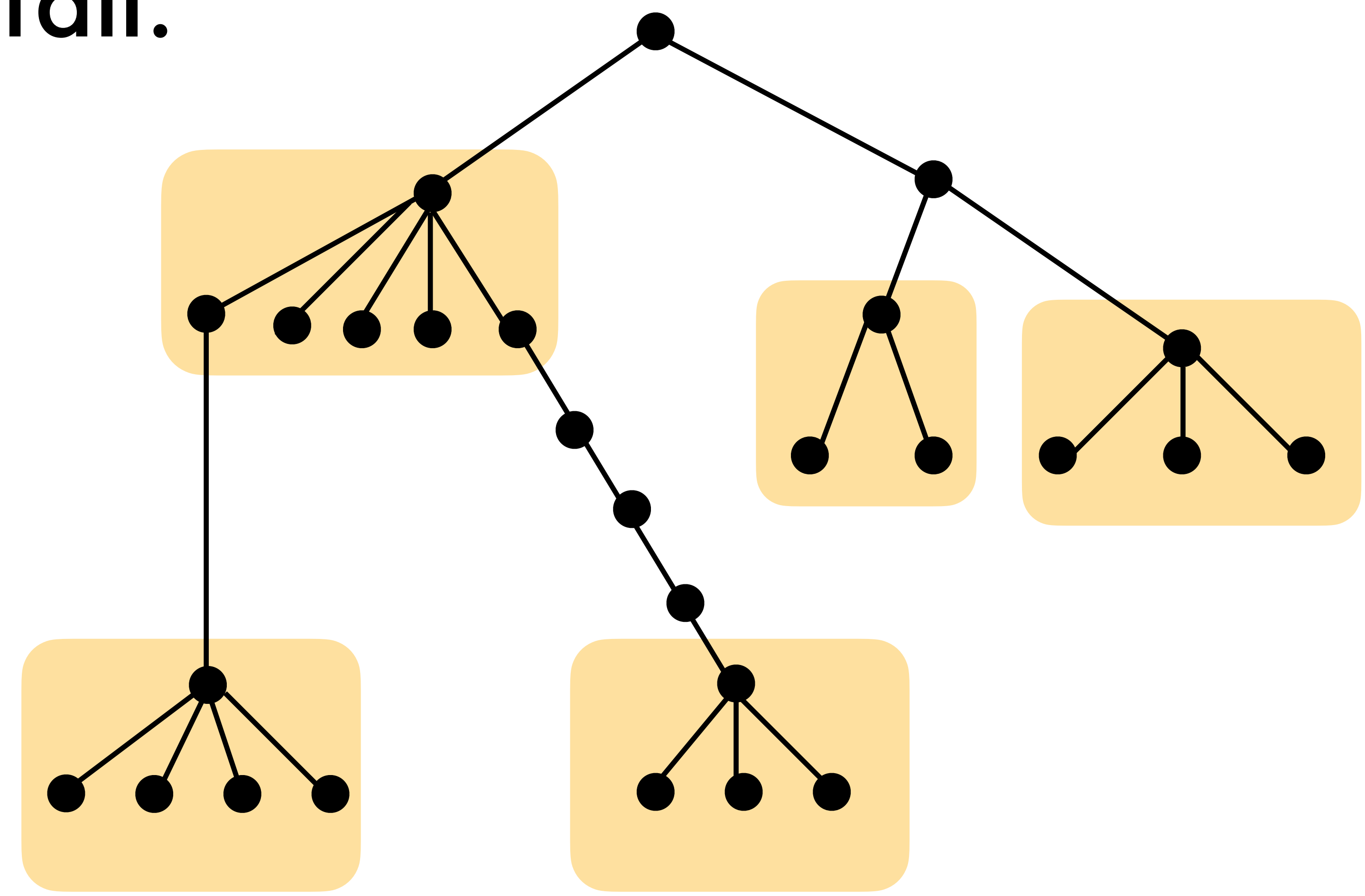


MLS on connected graphs admit a polynomial kernel - Proof

Greedy: Let us try to greedily build a solution and see where we fail.

Procedure: Let us try to construct vertex disjoint stars in G.

Let v be a vertex of degree at least 3.
Let S_v be a star with v and its neighbours in (the original graph G).
Remove $N^2(v)$ from G and repeat (as long as there is a vertex of degree at least 3 in the resulting graph).



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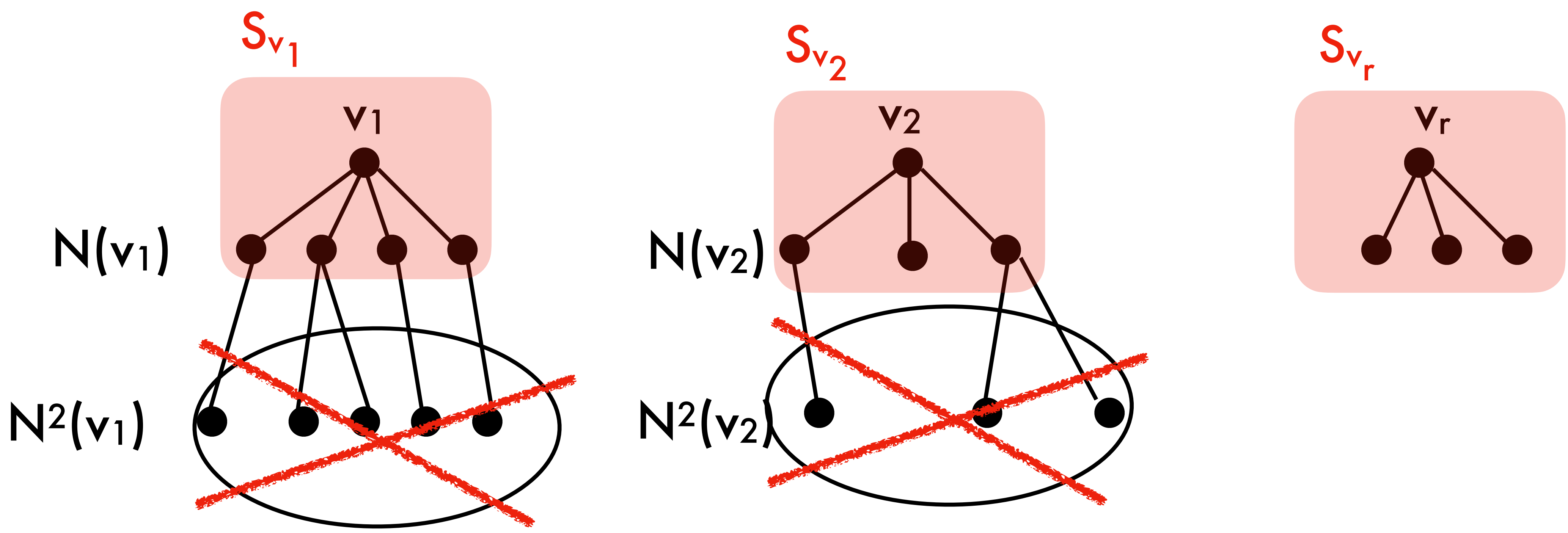

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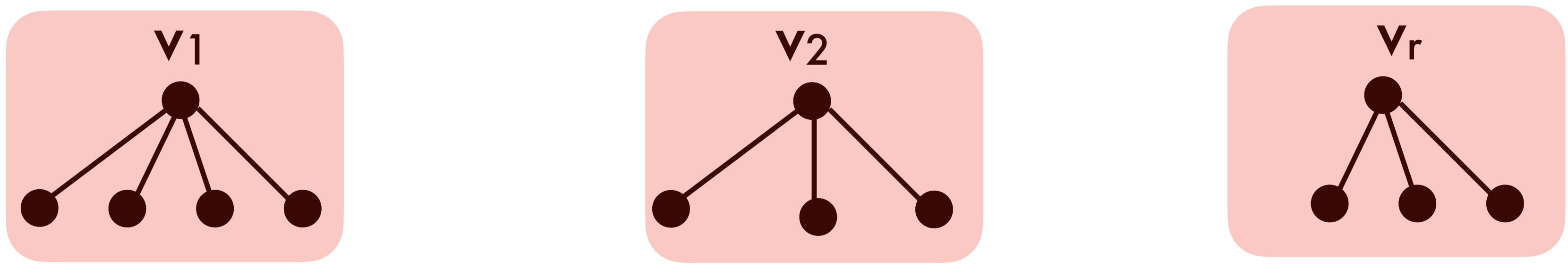

Constructing a subtree from these stars (with at least k leaves):
Join the red stars by adding arbitrary paths between the v_i vertices.
The resulting connected graph has at least r leaves.

MLS on connected graphs admit a polynomial kernel - Proof

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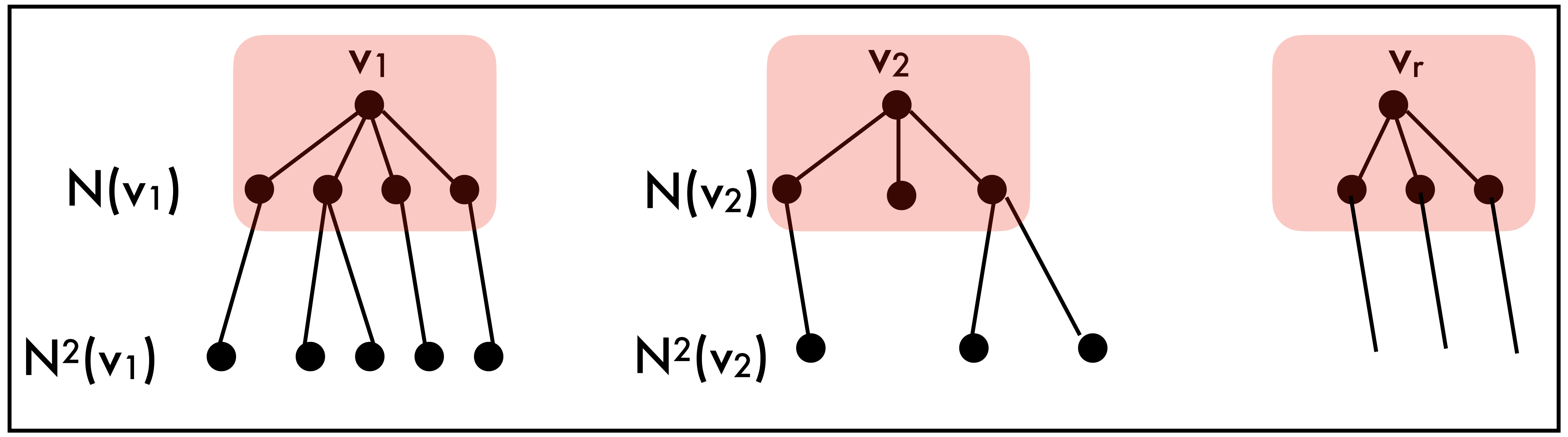
Suppose the above procedure runs for $r < k$ steps.



$|X| = O(k^2)$

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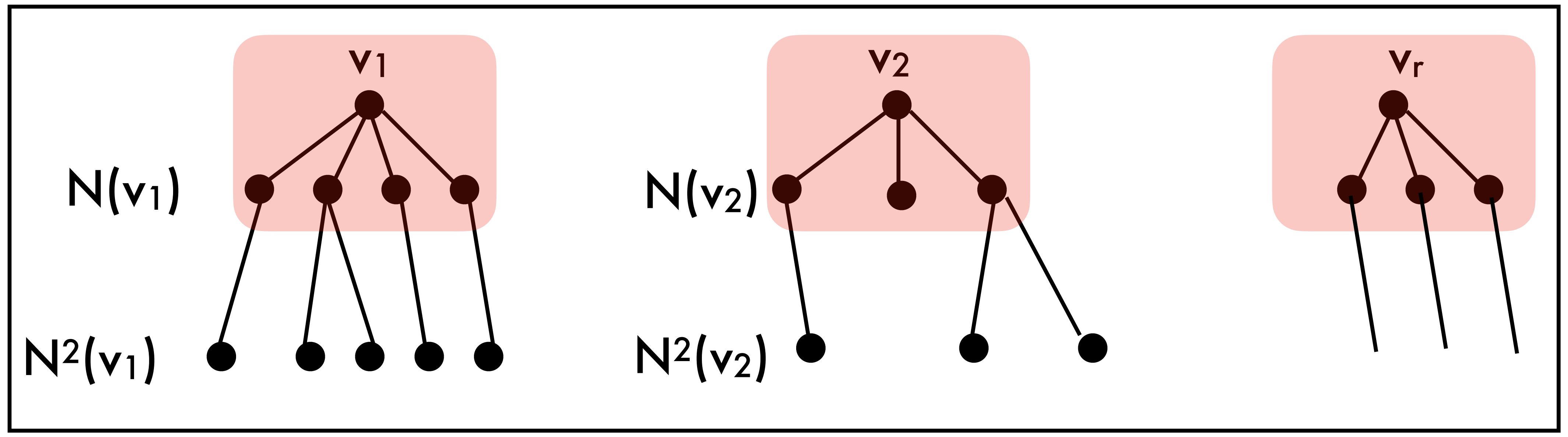

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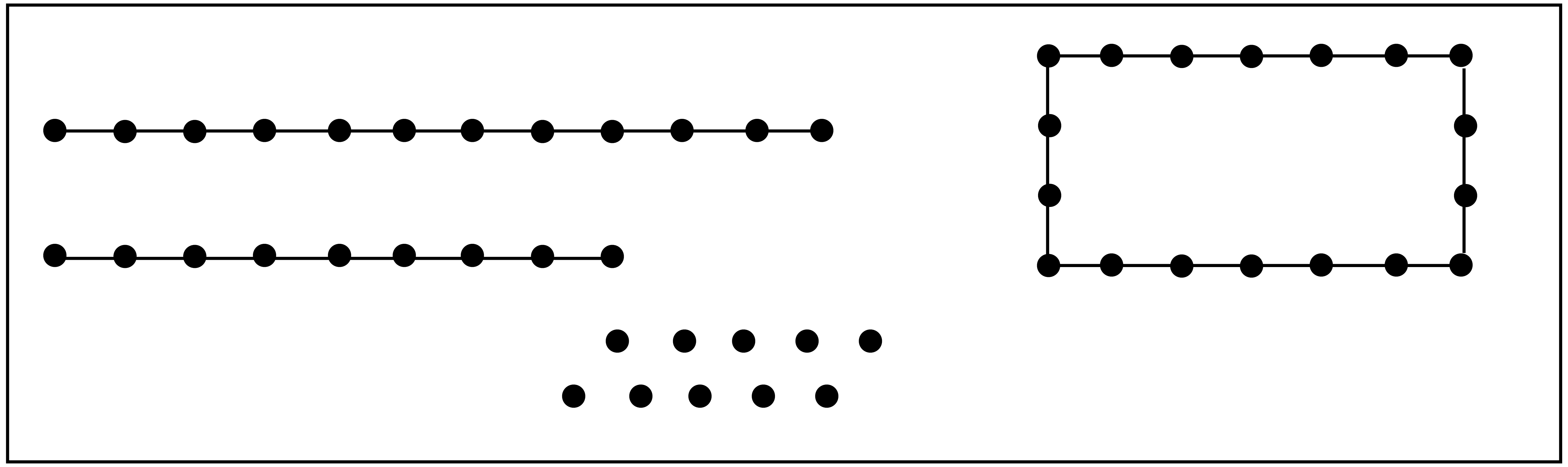

Every vertex of G-X has degree at most 2.
G-X is a disjoint union of paths and cycles.

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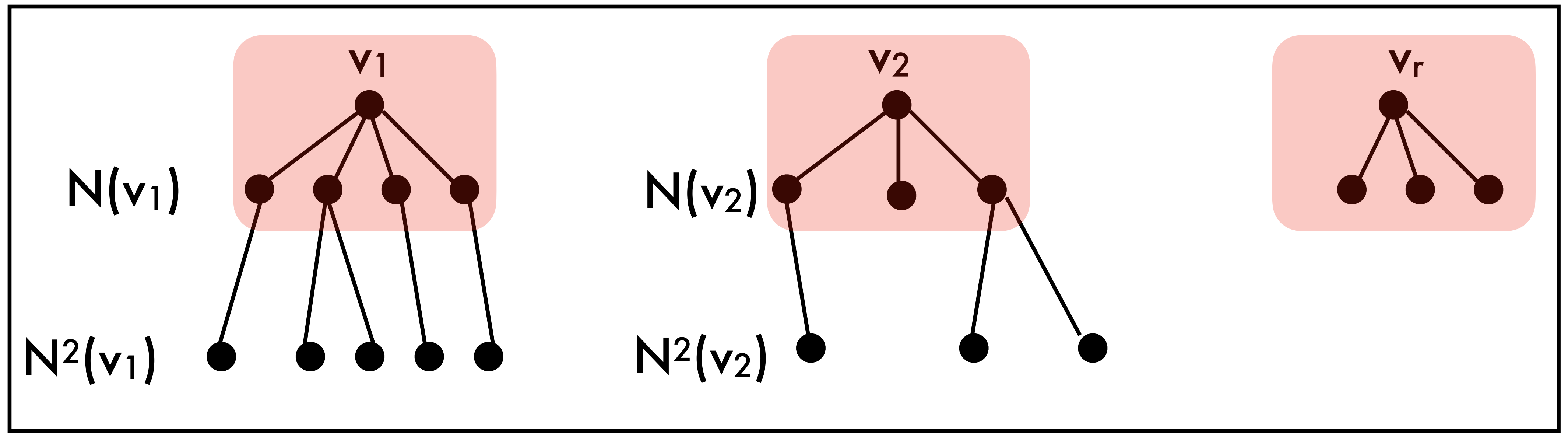

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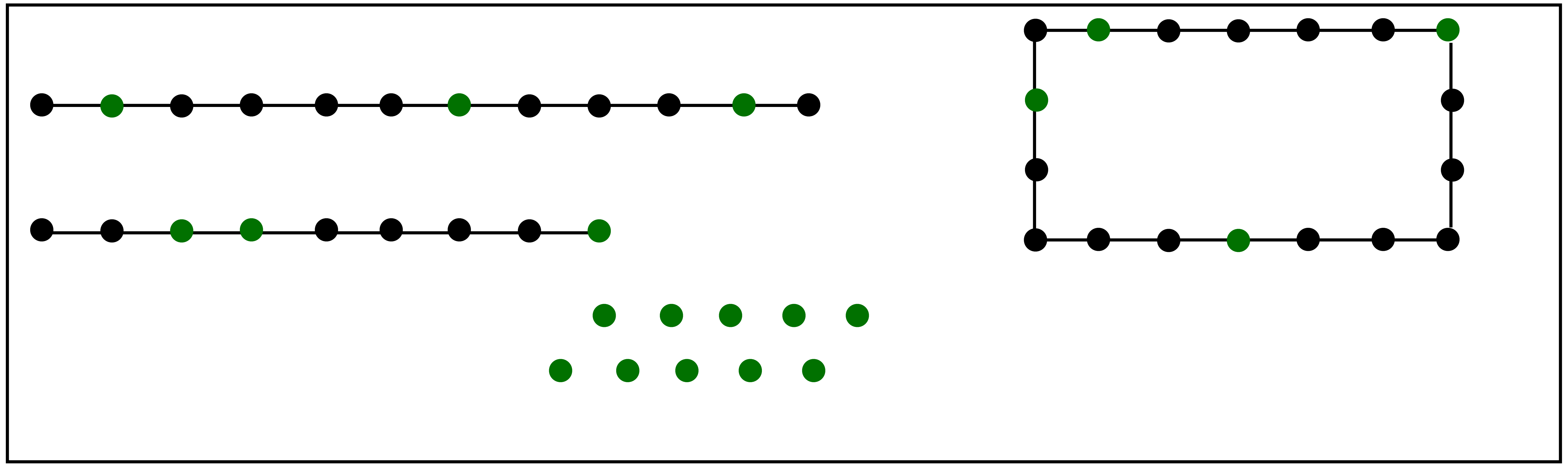

Every vertex of G-X has degree at most 2.
G-X is a disjoint union of paths and cycles.
The green vertices are neighbours of X.
$|N(X)| = O(k^2)$

MLS on connected graphs admit a polynomial kernel - Proof

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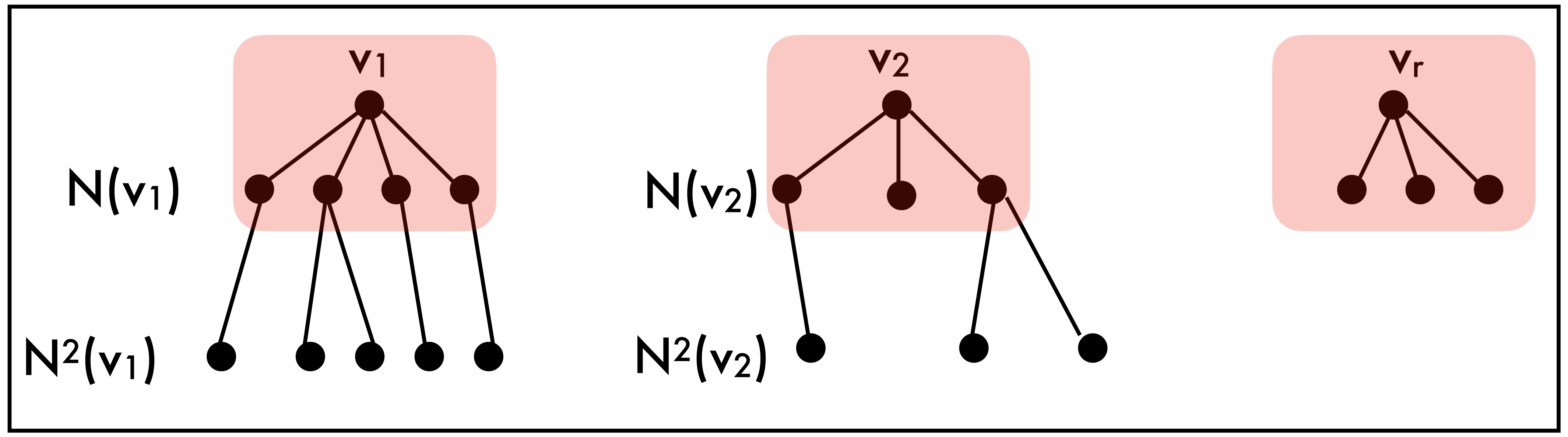

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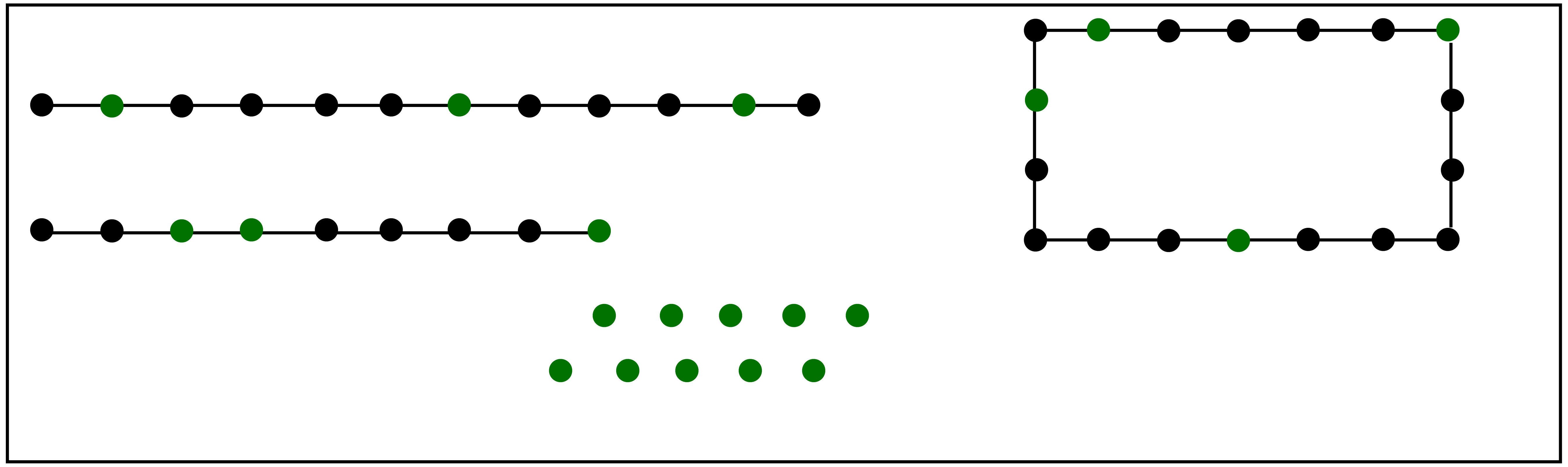

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$|N(X)| = O(k^2)$
The black vertices between two consecutive green vertices are degree 2 vertices in the entire graph.

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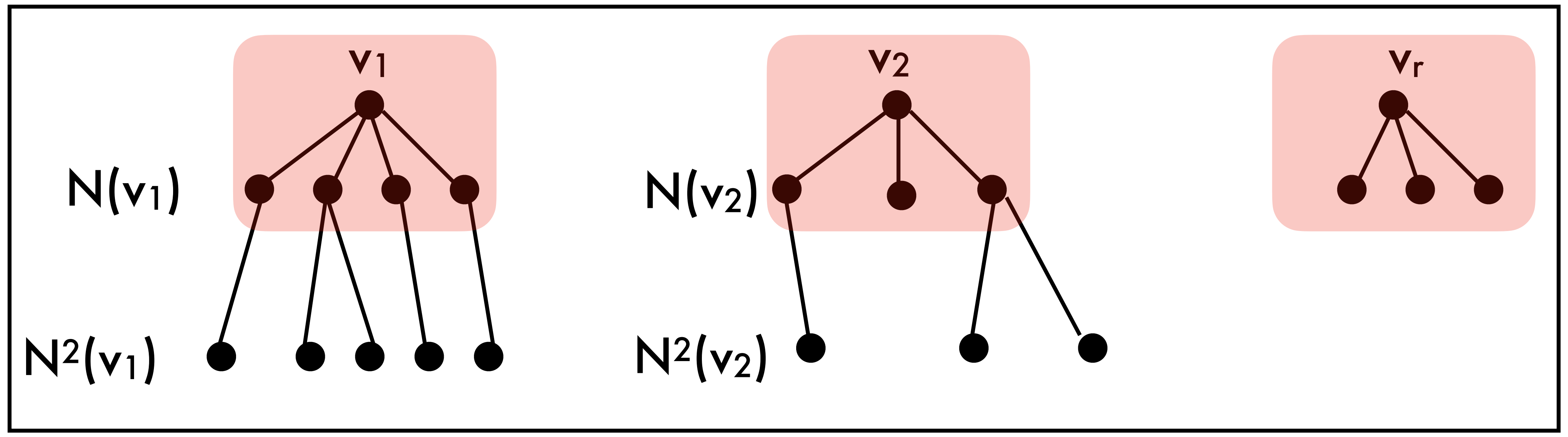

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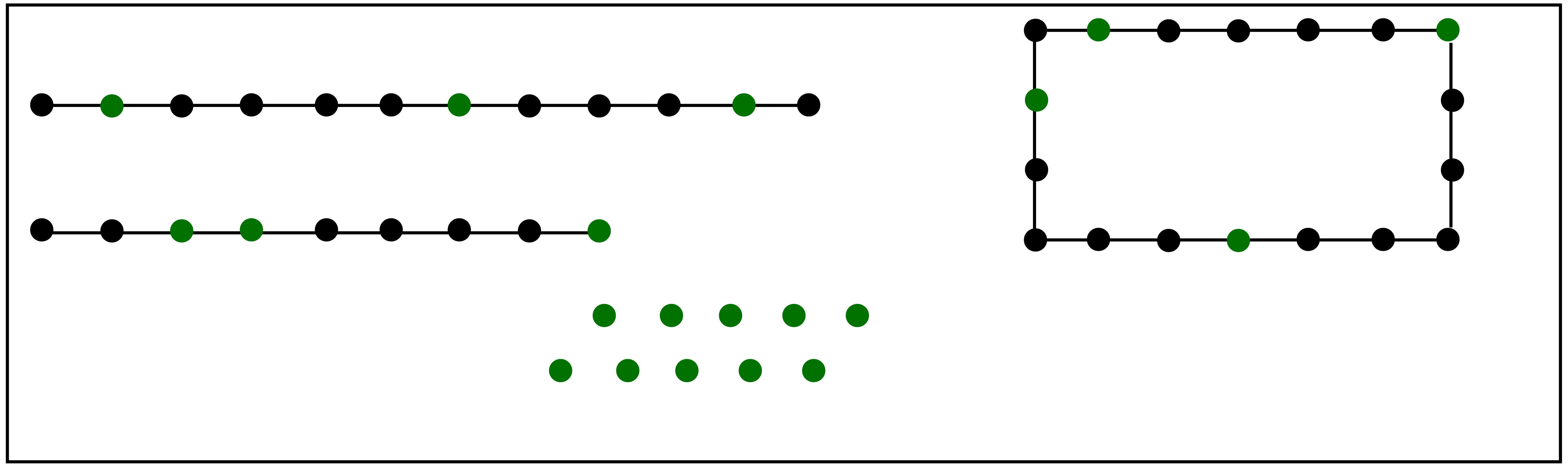

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G-X is a disjoint union of paths and cycles.
The green vertices are neighbours of X.
$|N(X)| = O(k^2)$
The black vertices between two consecutive green vertices are degree 2 vertices in the entire graph.
Therefore, $|V(G) \setminus X| \leq (|N(X)| + 1) = O(k^2)$

Lower bound machinery for Turing kernels?

- How to show that a problem does not exhibit any Turing kernel?
- So far, no machinery exists that allows one to prove such statements.
- Rather, we developed some hardness theory based on conjectures like,

Connected Vertex Cover does not admit a Turing kernel, or

Steiner Tree does not admit a Turing kernel.