Kernelization lower bounds

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No answer until 2008.

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Decision problems as languages

- Un-parameterized problems can be viewed as languages for a finite alphabet Σ.
- A language L over Σ is a subset of strings in Σ^*.
- A Parameterized problem is a set of pairs (x, k), where $x \in \Sigma^*$ and k is a non-negative integer.
- In the unparameterized version of a parameterized problem k is appended at the end of input in unary.

Kernelization algorithm for a language L

This is a polynomial time algorithm that takes as input (x, k) and outputs (x', k') such that

$$(x, k) \in L \text{ if and only if } (x', k') \in L, \text{ and}$$
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Let L, R be parameterized problems.

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- We say L has no polynomial compression if L has no polynomial compression to any problem R.
- We assume nothing about the problem R.
- If $R \in$ NP and L is NP-hard, then a polynomial compression from L to R implies a polynomial kernel for L.

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Intuition (with example)

k-Path

Input: A graph G and a positive integer k
Question: Does there exist a path of length k in G?

Suppose (for the sake of contradiction) k-Path admits a polynomial kernel, say with k^3 vertices.

- Let $(G_1, k), (G_2, k), \ldots, (G_t, k)$ be $t = k^7$ instances of k-Path.
- Create another instance (H, k) of k-Path where H is the disjoint union of the graphs $G_1, \ldots, G_t$.
- H has a path of length k if and only if G_i has a path of length k for some $i \in \{1, \ldots, t\}$ ("H is an OR of G_is").
- Kernelize (H, k) and get (H', k') where $|V(H')| \leq k^3$. Bit size of the reduced instance (H', k') is $\approx k^6$.

The bit size of the reduced instance is strictly less than the number of instances (t) that we started with!

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No accountability for the loss of information!

Loss of important information

In polynomial time, we **completely forgot** about some of the (t) input instances.
How did this polynomial time algorithm knew that they could be discarded (that they did not have k-paths)? This is unlikely to have happened for any NP-hard problem!

OR-distillation

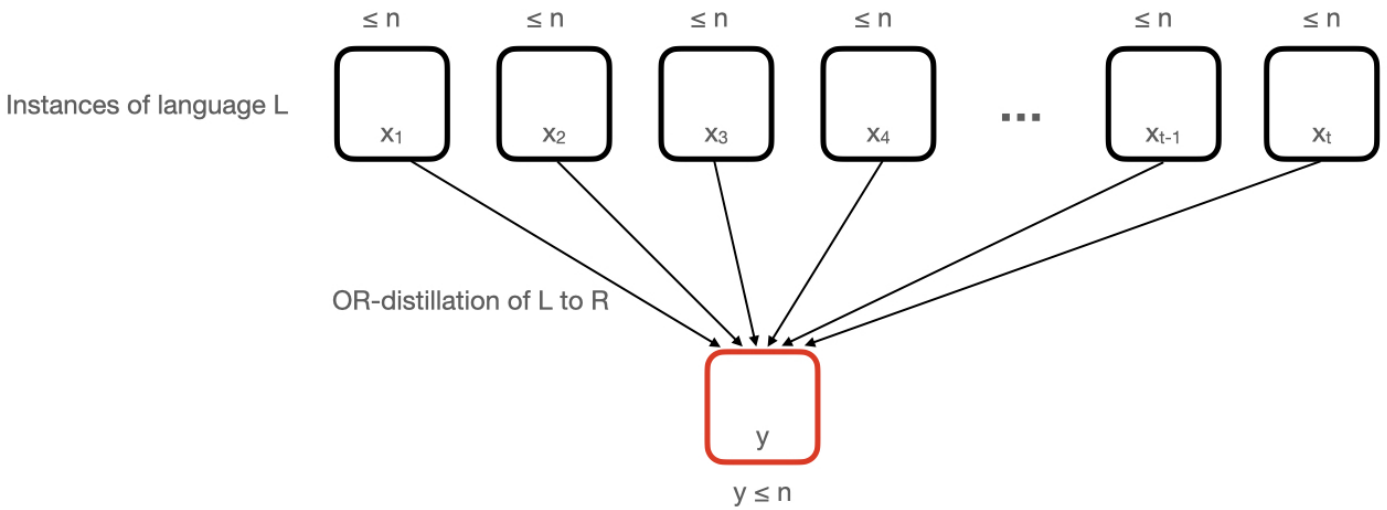
Let L, R be un-parameterized languages.

OR-distillation of L into R

Input: Strings $x_1, \ldots, x_t$ each of length at most n.
Output: A string y such that:

- $y \in R$ if and only if $x_i \in L$ for some $i \in \{1, \ldots, t\}$, and
- $|y| = poly(\max |x_i|)$.

Time: polynomial in the input, that is $poly(\sum_{i=1}^{t} |x_i|)$.



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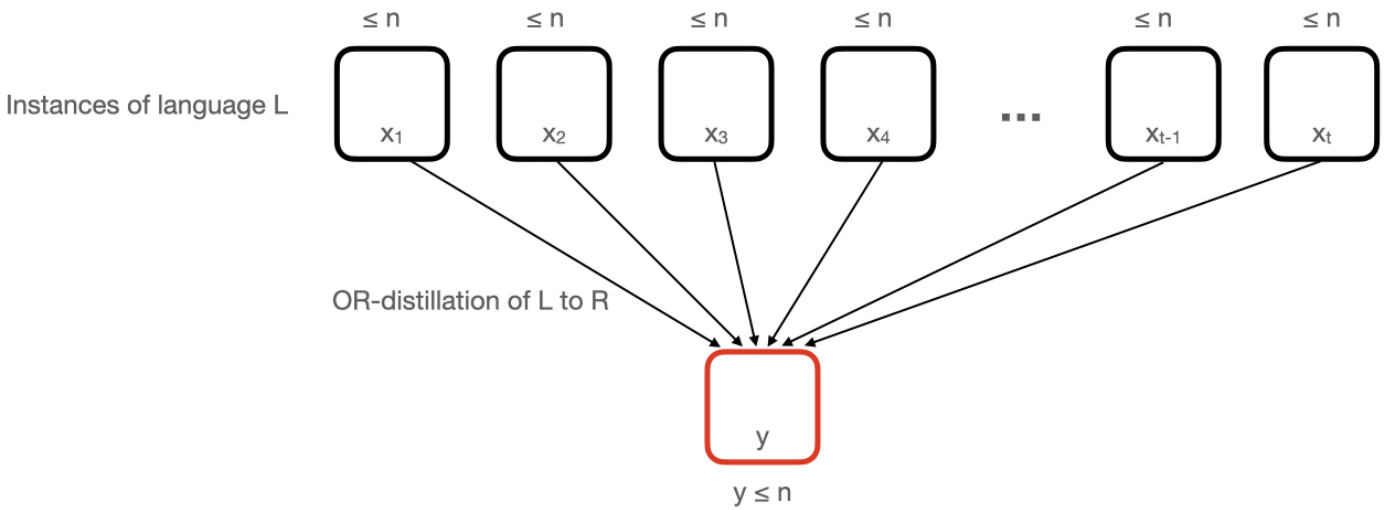

OR-distillation

- Define another language, OR-$L = \{x_1\# \ldots \#x_t : x_i \in L \text{ for some } i\}$.
- OR-distillation from L into R is a polynomial compression from OR-$L/\max|x_i|$ to R.

The main theorem (without proof)

OR-distillation theorem [Fortnow, Santhanam 2008]

No NP-hard problem admits an OR-distillation into any language R, unless NP$\subseteq$coNP/poly.

- Intuitive meaning of NP$\subseteq$coNP/poly: Verifying proofs in polynomial time cannot be turned into verifying counterexamples in polynomial time, even if we allow polynomial advice.
- Implication of NP$\subseteq$coNP/poly: PH$=\Sigma_3^P$, that is the polynomial hierarchy collapses to the third level.
- This is believed to be highly unlikely (not as unlikely as P=NP but still highly unlikely).

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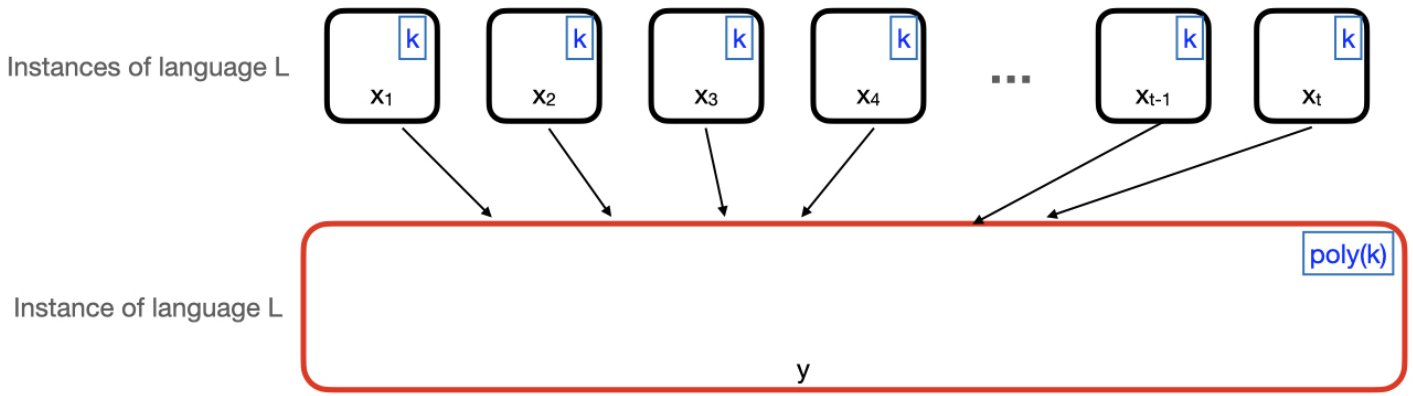

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If there exists an OR-composition for a parameterized problem L such that the un-parameterized version of L is NP-hard, then L does not admit a polynomial kernel unless $\mathrm{NP} \subseteq \mathrm{coNP/poly}$.

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OR-composition theorem: Proof

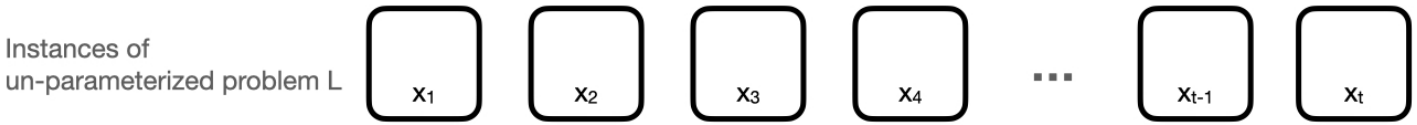

Figure: OR-distillation of L into OR-L

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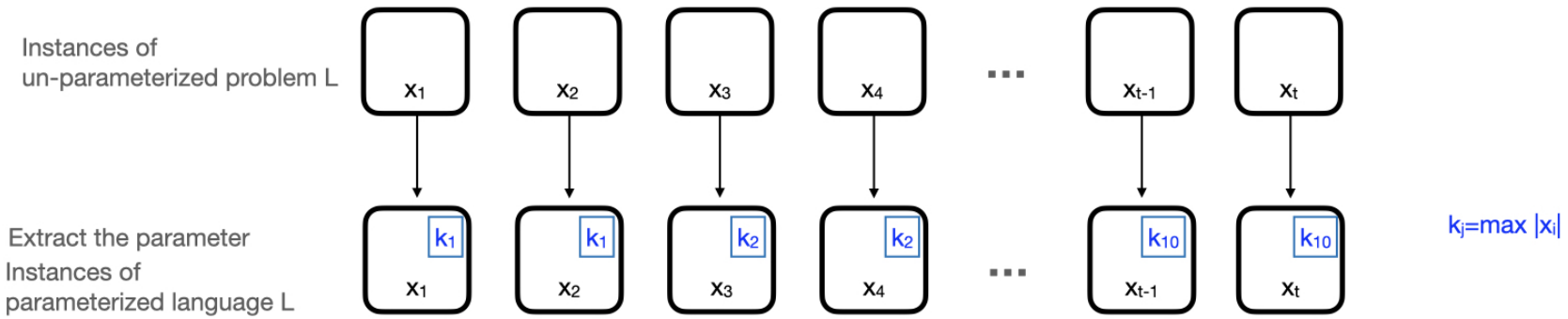

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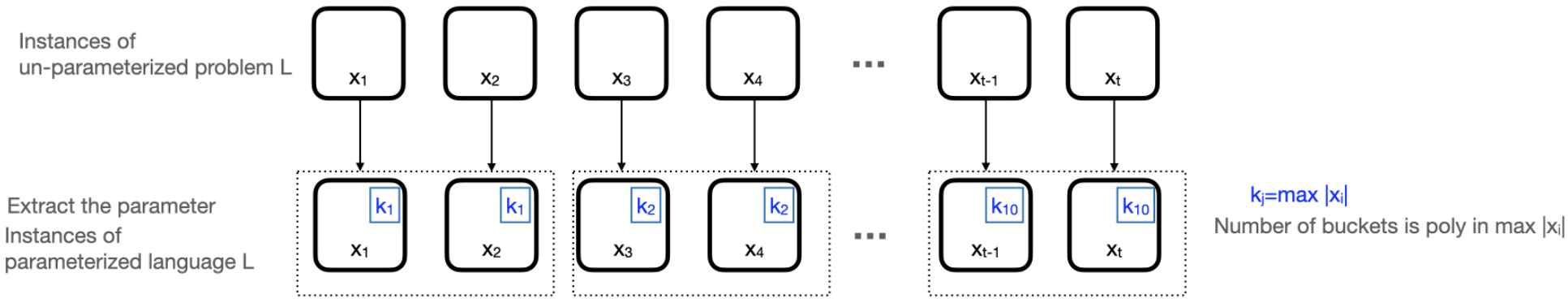

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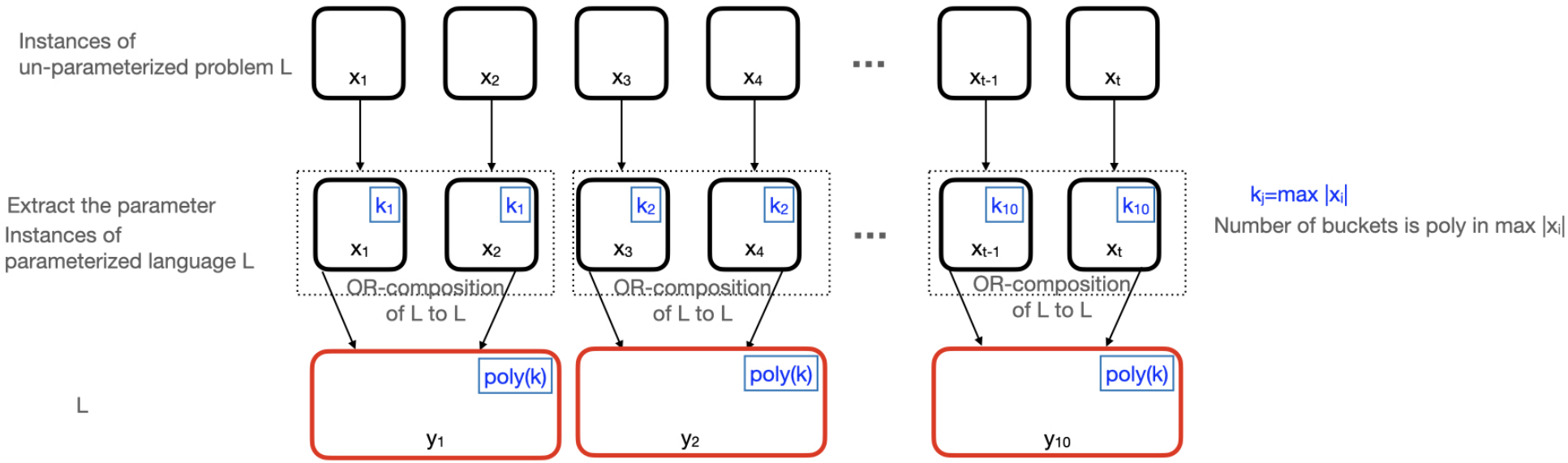

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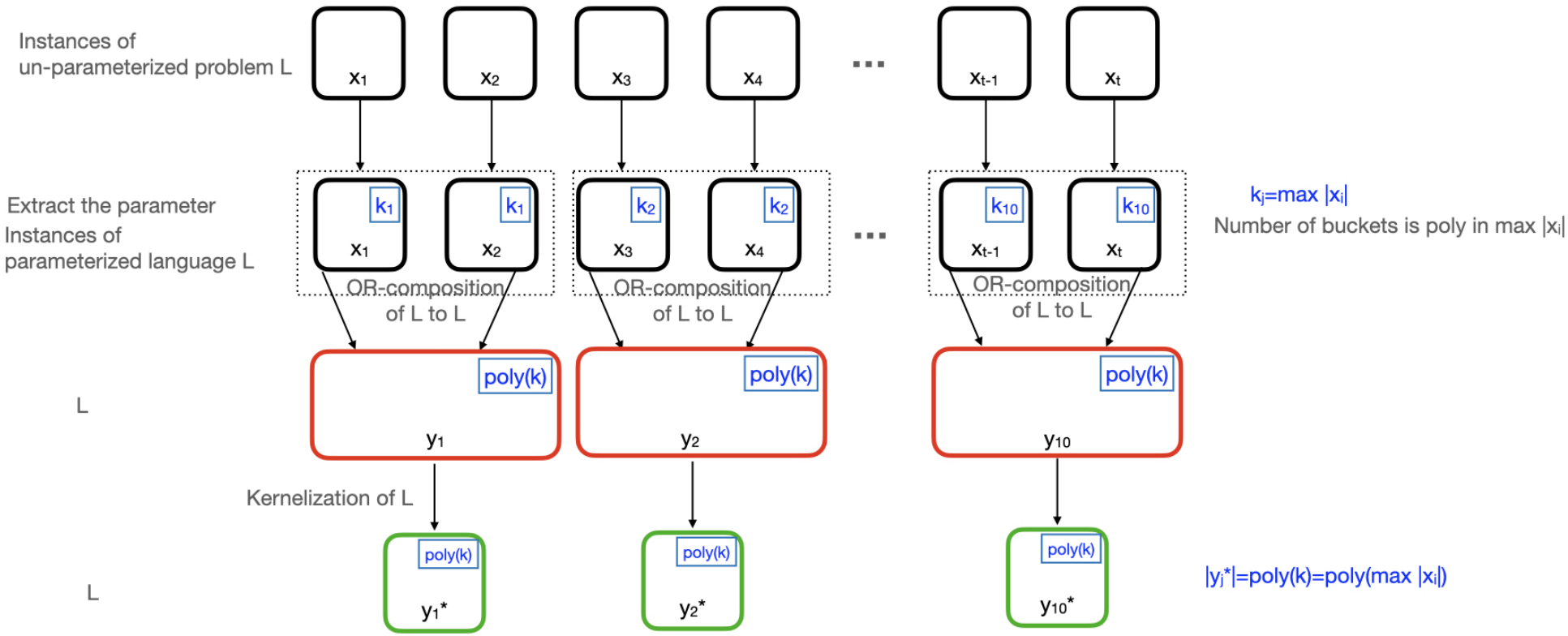

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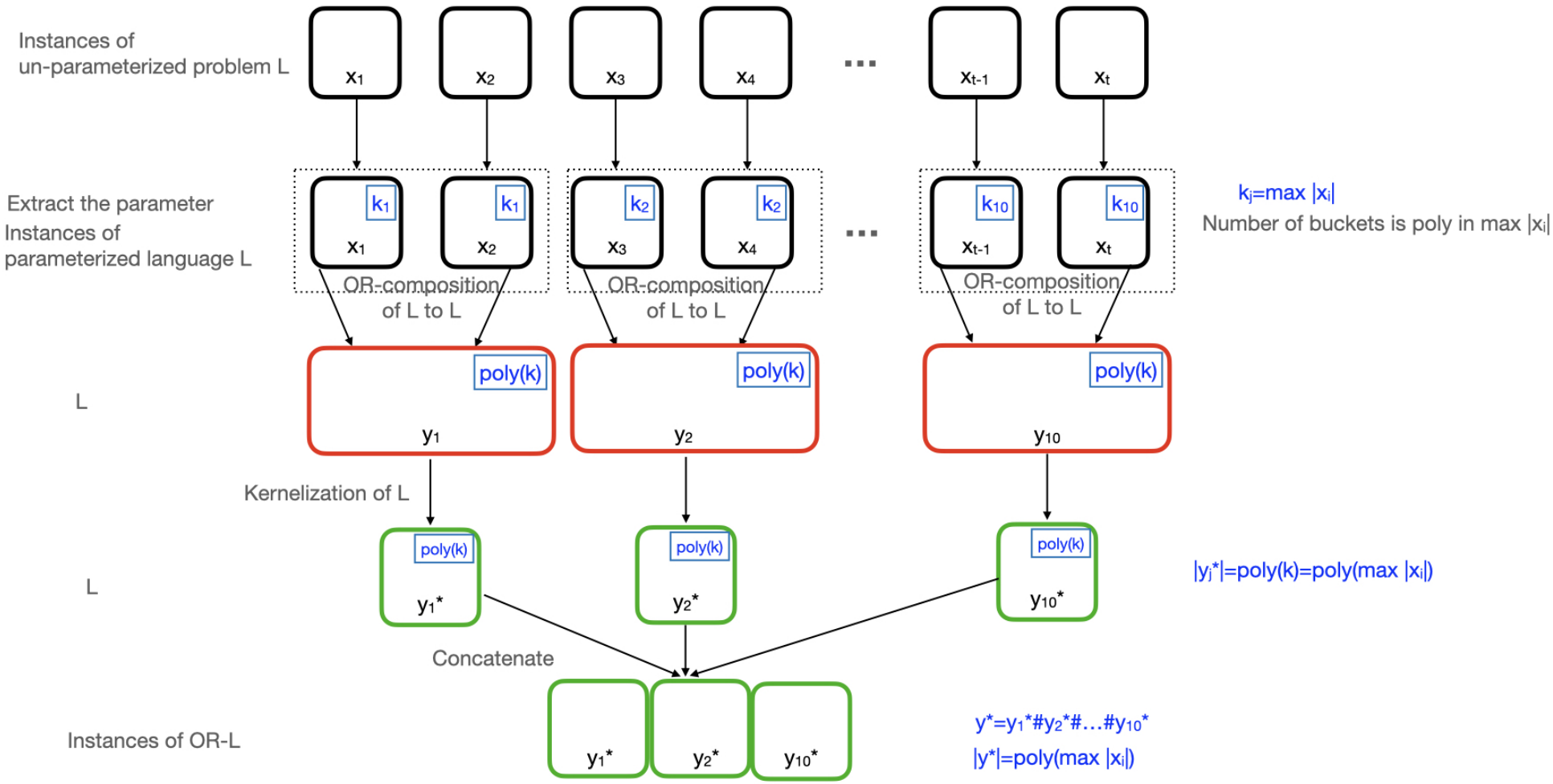

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Application of the OR-composition theorem

Theorem

k-Path does not admit a polynomial kernel, unless NP$\subseteq$coNP/poly.

Since k-Path is an NP-hard problem, for the proof of the theorem it is enough to give an OR-composition for k-Path.

OR-composition for k-Path: Given $(G_1, k), \ldots, (G_t, k)$, output $(H = (G_1 \uplus \ldots \uplus G_t), k)$.

Correctness follows because a graph has a path of length k if and only if at least one of its connected components has it.

Show that k-Cycle does not admit a polynomial kernel, unless NP$\subseteq$coNP/poly.

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Enhancing the framework for compositionality

We have an OR-composition for an NP-hard problem L (from L to L).

1. Can we show that L has no polynomial compression to any language R?
   - Yes.
2. What happens if we give an OR-composition from Q to L (instead of L to L)? Can we still say no polynomial compression for L?
   - Yes, as long as Q is an NP-hard problem.
3. Can we do more refined bucketting? Eg. Can we ensure that there are same number of vertices in all the graphs of the same bucket?
   - Yes, as long as number of buckets is polynomial.
4. Can we say $k^* = poly(k + \max |x_i|)$?
   - Yes.
5. How large can t be?
   - If $t > \Sigma^{n+1}$, where $n = \max |x_i|$, then we can remove the duplicate input instances.
   - Therefore, $\log t = \mathcal{O}(n)$.
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Enhancing the framework for compositionality

We have an OR-composition for an NP-hard problem L (from L to L).

1. Can we show that L has no polynomial compression to any language R?
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OR-cross composition [Bodlaender, Jansen, Kratsch 2011]

Polynomial equivalence relation

A polynomial equivalence relation is an equivalence relation $\equiv$ defined over strings in Σ^* such that:

- for any two strings $x, y \in \Sigma^*$, one can check whether $x \equiv y$ in $poly(|x| + |y|)$ time, and
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Some common examples of polynomial equivalence relations include:

- partitioning the words/graphs such that each equivalence class has words/graphs of same length/number of vertices, or
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OR-cross composition [Bodlaender, Jansen, Kratsch 2011]

Let Q be an unparameterized problem and L be a parameterized problem

OR-cross composition from Q to L

Input: $x_1, \ldots, x_t$, such that there exists a polynomial equivalence relation $\equiv$ where $x_1, \ldots, x_t$ belong to the same equivalence class of $\equiv$.

Output: (y, k^*) such that:

- $k^* = poly(\log t + \max_{i=1}^{t} |x_i|)$, and
- $(y, k^*) \in L$ if and only if $x_i \in Q$ for some i.

Time: $poly(\sum_{i=1}^{t} |x_i|)$.

OR-cross composition theorem

If there is a OR-cross composition from Q to L and Q is NP-hard, then L has no polynomial compression, unless NP$\subseteq$coNP/poly.

Remark: This notion is particularly useful in refuting polynomial kernels with structural parameterizations, for example, Clique/vertex cover.

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k-Path Packing

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Input: A graph G and an integer k
Question: Does there exist a collection of k mutually vertex disjoint paths of length k each?
Parameter: k

Does k-Path Packing admit a polynomial kernel/compression?

- Give a reduction from k-Path (G, k) to k-Path Packing (H, k) as follows: H is the disjoint union of G and $k-1$ vertex disjoint paths of length k each.
- G has a path of length k if and only if H has a k-path packing.
- If k-Path Packing admits a polynomial kernel/compression then so does k-path.

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Polynomial parameter transformations (PPT)

Let L, R be parameterized problems.

Polynomial parameter transforamtions from L to R

We say there is a polynomial parameter transformation from L to R if there exists a polynomial time algorithm that takes as input (x, k) and outputs (y, k') such that:

- $(x, k) \in L$ if and only if $(y, k') \in R$, and
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PPT Theorem

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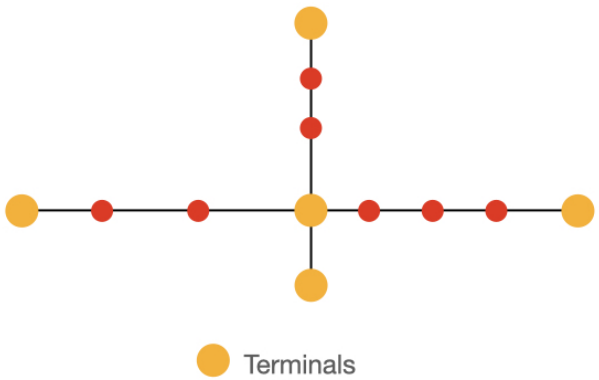
Application: Steiner Tree

Steiner Tree

Input: A graph G, a set of terminals $K \subseteq V(G)$, a positive integer k.
Question: Does there exist a connected subgraph (tree) of G on size at most k vertices that contains all the terminal vertices?
Parameter: k

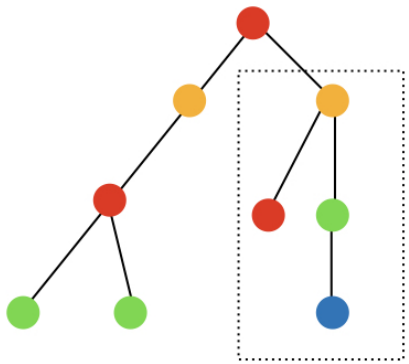


Colorful Graph Motif

Colorful Graph Motif (CGM)

Input: A graph G, a coloring function $c : V(G) \to \{1, \ldots, k\}$.
Question: Does there exist a connected subgraph of G that contains exactly one vertex of each color?
Parameter: k

CGM is NP-hard even on trees.



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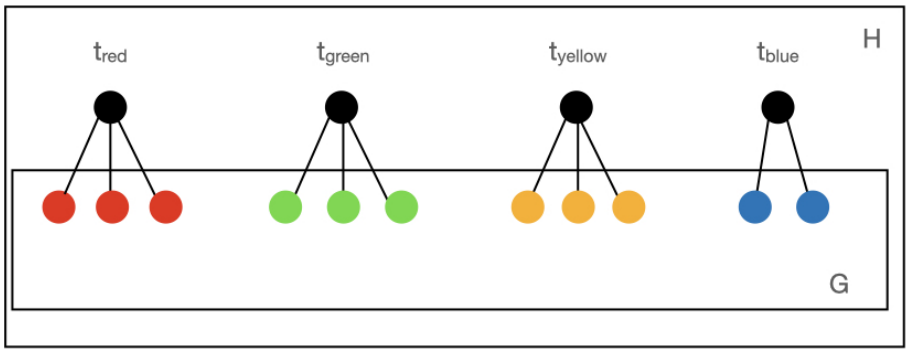
PPT from CGM to Steiner Tree

Let (G, c, k) be an instance of Colorful Graph Motif.
Construct H from G by adding a new terminal vertex for each color class and making it adjacent to the respective color class.



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Claim

(G, c, k) is a Yes-instance of Colorful Graph Motif if and only if $(H, K = \{t_{red}, t_{green}, t_{yellow}, t_{blue}, ..\}, 2k)$ is a Yes-instance of Steiner Tree.

Theorem

Steiner Tree does not admit a polynomial kernel/compression parameterized by k, unless NP$\subseteq$coNP/poly.

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Weak compositions and stronger lower bounds

One can further refine the notion of OR-cross compositions to define weak compositions which give tight polynomial lower bounds for kernel sizes.

For example, they can be used to show the following:

- Vertex Cover does not admit a polynomial compression with bit size $\mathcal{O}(k^{2-\epsilon})$, unless NP$\subseteq$coNP/poly.
- Feedback Vertex Set does not admit a polynomial compression with bit size $\mathcal{O}(k^{2-\epsilon})$, unless NP$\subseteq$coNP/poly.
- d-Hitting Set and d-Set Packing parameterized by $|U|$ does not admit a polynomial compression with bit size $\mathcal{O}(|U|^{d-\epsilon})$, unless NP$\subseteq$coNP/poly.
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