Parameterized Algorithms

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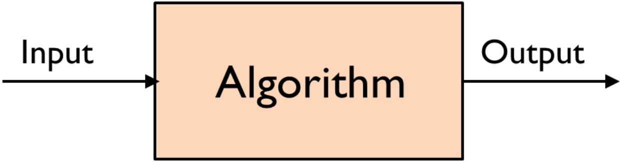
(tutorials)

Lecture #1
October 19, 2021

Modalities

- Lectures every Tuesday (10:15-12:00)
- Exercises sheets handed out every ≈ 2 weeks, needed to be submitted in ≈ 1 week
- Tutorials to discuss the exercises (dates to be discussed)
- Oral exams
- Deadline for unregistering from the course: first oral exam

Prerequisites: algorithms



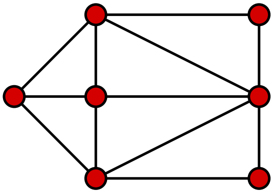
- Worst-case analysis: guaranteed running time $T(n)$ for every input of size n.
- Big-O notation: hiding constant factors + ignoring small inputs.
- Two main classes:
  - Polynomial time ($O(n)$, $O(n \log n)$, $O(n^2)$, $\ldots$)

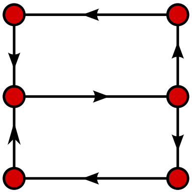
    Example: Quick Sort, Matrix multiplication, Perfect Matching
  - Exponential time (2^n, $2^{\sqrt{n}}$, $n!$, $\ldots$)

    Example: brute force search, dynamic programming for TSP

Prerequisites: graphs

- Graph G with set of vertices $V(G)$ and set of edges $E(G)$.
- Directed graphs: each edge $\overrightarrow{uv}$ has an orientation.
- Basic graph-theoretic terms: degree of a vertex, connectedness, (induced) subgraphs, planar graphs, proper coloring of the vertices of a graph, clique, independent set, matching.





Classic algorithmic problems on graphs: connectivity, shortest paths, perfect matching, maximum flow, minimum $s-t$ cut, ...

Prerequisites: graph classes

Some important classes:

- Regular graphs: every vertex has the same degree
- Trees, forests
- Planar graphs
- Intersection graphs (e.g., interval graphs)

Prerequisites: optimization

- **Decision problems:** return a yes-no answer
- **Search problems:** return a solution
- **Optimization problems:** return the best solution
  - feasible solution
  - cost function
  - goal is to minimize/maximize cost

Classic optimization problems: shortest path, maximum flow, linear programming, bin packing, knapsack, ...

Turning an optimization problem into a decision problem:
"Is there a solution with cost at least/at most k?"

Prerequisites: computational complexity

A brief review:

- We usually aim for **polynomial-time** algorithms: the worst-case running time is $O(n^c)$, where n is the input size and c is a constant.
- Classical polynomial-time algorithms: shortest path, perfect matching, minimum spanning tree, 2SAT, convex hull, planar drawing, linear programming, etc.
- It is unlikely that polynomial-time algorithms exist for **NP-hard** problems.
- Unfortunately, many problems of interest are NP-hard: Hamiltonian Cycle, 3-Coloring, 3SAT, etc.
- We expect that these problems can be solved only in exponential time (i.e., $O(c^n)$).

Can we say anything nontrivial about NP-hard problems?

Parameterized problems

Main idea

Instead of expressing the running time as a function $T(n)$ of n, we express it as a function $T(n, k)$ of the input size n and some parameter k of the input.

In other words: we do not want to be efficient on all inputs of size n, only for those where k is small.

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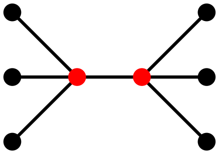
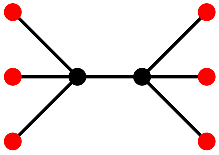
In other words: we do not want to be efficient on all inputs of size n, only for those where k is small.

What can be the parameter k?

- The size k of the solution we are looking for.
- The maximum degree Δ of the input graph.
- The dimension d of the point set in the input.
- The length L of the strings in the input.
- The length ℓ of clauses in the input Boolean formula.
- $\ldots$

Parameterized complexity

Problem:	VERTEX COVER	INDEPENDENT SET
Input:	Graph G, integer k	Graph G, integer k
Question:	Is it possible to cover the edges with k vertices?	Is it possible to find k independent vertices?
		
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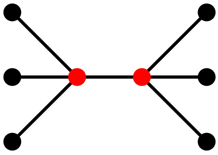

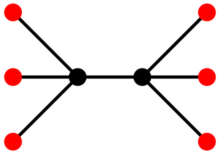

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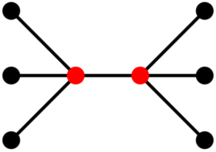

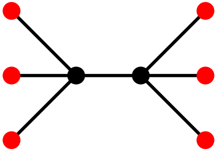

Complexity:	NP-complete	NP-complete
Brute force:	$O(n^k)$ possibilities	$O(n^k)$ possibilities
	$O(2^k n^2)$ **algorithm exists**	**No** $n^{o(k)}$ **algorithm known**

Bounded search tree method

Algorithm for Vertex Cover:

$e_1 = u_1 v_1$

Bounded search tree method

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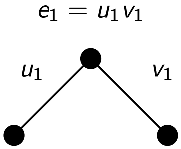

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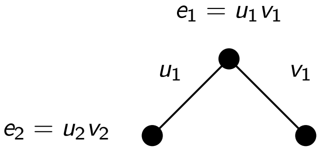

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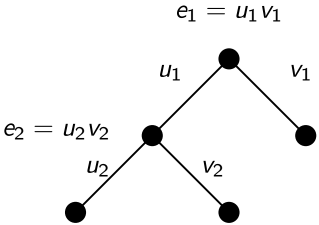

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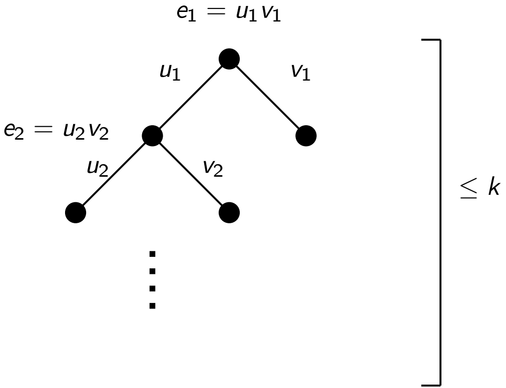

Height of the search tree $\leq k \Rightarrow$ at most 2^k leaves $\Rightarrow 2^k \cdot n^{O(1)}$ time algorithm.

Fixed-parameter tractability

Main definition

A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant c.

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A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant c.

Examples of NP-hard problems that are FPT:

- Finding a vertex cover of size k.
- Finding a path of length k.
- Finding k disjoint triangles.
- Drawing the graph in the plane with k edge crossings.
- Finding disjoint paths that connect k pairs of points.
- ...

More formally

- We consider only **decision problems** here.
- Let Σ be a finite alphabet used to encode the inputs
  - ($\Sigma = \{0, 1\}$ for binary encodings)

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- A **parameterized problem** is a set $P \subseteq \Sigma^* \times \mathbb{N}$
  - $P = \{(x_1, k_1), (x_2, k_2), \dots\}$
- The set P contain the tuples (x, k) where the answer to the question encoded by (x, k) is yes; k is the **parameter**

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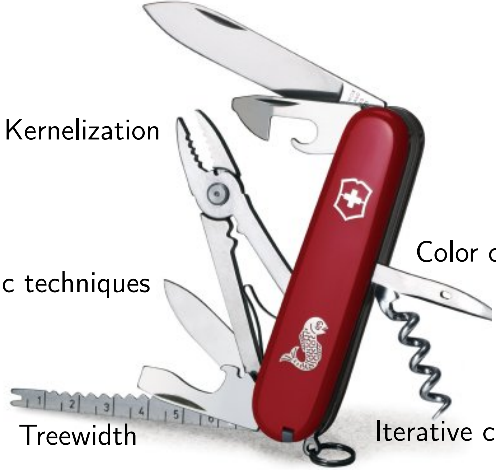
- The set P contain the tuples (x, k) where the answer to the question encoded by (x, k) is yes; k is the **parameter**
- A parameterized problem P is **fixed-parameter tractable** if there is an algorithm that, given an input (x, k)
  - decides if (x, k) belongs to P or not, and
  - the running time is $f(k)n^c$ for some computable function f and constant c.

FPT techniques



W[1]-hardness

Negative evidence similar to NP-completeness. If a problem is **W[1]-hard,** then the problem is not FPT unless FPT=W[1].

Some W[1]-hard problems:

- Finding a clique/independent set of size k.
- Finding a dominating set of size k.
- Finding k pairwise disjoint sets.
- ...

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General principle of hardness

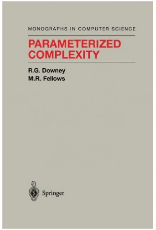
With an appropriate **reduction** from k-Clique to problem P, we show that if problem P is FPT, then k-Clique is also FPT.

Parameterized complexity

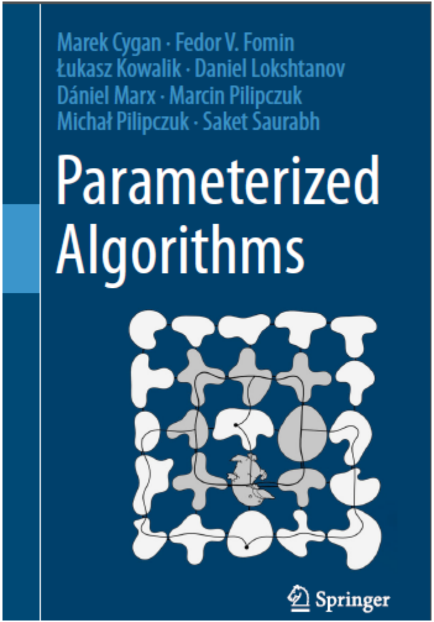

Rod G. Downey
Michael R. Fellows

Parameterized Complexity

Springer 1999



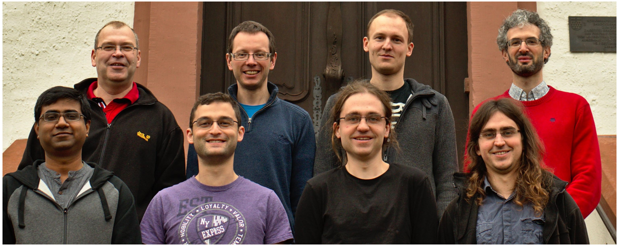
- The study of parameterized complexity was initiated by Downey and Fellows in the early 90s.
- First monograph in 1999.
- By now, strong presence in most algorithmic conferences.



Parameterized Algorithms

Marek Cygan, Fedor V. Fomin,
Lukasz Kowalik, Daniel Lokshtanov,
Dániel Marx, Marcin Pilipczuk,
Michał Pilipczuk, Saket Saurabh

Springer 2015



Course outline

- Basic techniques
  - bounded search trees
  - color coding
  - dynamic programming
  - iterative compression
- Complexity
- Kernelization
- Treewidth
- Advanced topics:
  - cuts and separators
  - matroids
  - algebraic techniques

Bounded search tree method

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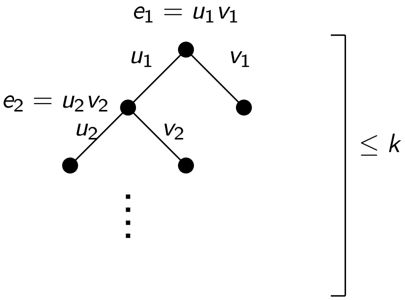
Algorithm for VERTEX COVER

- **Main idea:** reduce problem instance (x, k) to solving a bounded number of instances with parameter $< k$.
- We should be able to solve instance (x, k) in polynomial time using the solutions of the new instances.
- If the parameter strictly decreases in every recursive call, then the depth is at most k.
- Size of the search tree:
  - If we branch into c directions: c^k.
  - If we branch into $O(k)$ directions: $k^{O(k)} = 2^{O(k \log k)}$.
  - (If we branch into $O(\log n)$ directions: $O(n) + 2^{O(k \log k)}$.)



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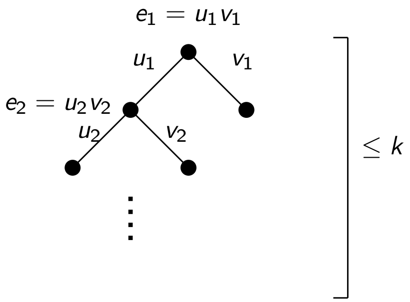

Next: A $1.41^k \cdot n^{O(1)}$ time algorithm for VERTEX COVER.

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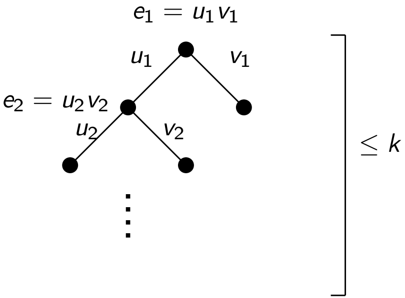

Next: A $O^*(1.41^k)$ time algorithm for VERTEX COVER.

Improved branching for Vertex Cover

- If every vertex has degree ≤ 2, then the problem can be solved in polynomial time.
- **Branching rule:**
  If there is a vertex v with at least 3 neighbors, then
  - either v is in the solution,
  - or every neighbor of v is in the solution.

Crude upper bound: $O^*(2^k)$, since the branching rule decreases the parameter.

Improved branching for Vertex Cover

- If every vertex has degree ≤ 2, then the problem can be solved in polynomial time.
- **Branching rule:**
  If there is a vertex v with at least 3 neighbors, then
  - either v is in the solution, $\Rightarrow$ k decreases by 1
  - or every neighbor of v is in the solution. $\Rightarrow$ k decreases by at least 3

Crude upper bound: $O^*(2^k)$, since the branching rule decreases the parameter.

But it is somewhat better than that, since in the second branch, the parameter decreases by at least 3.

Better analysis

Let $T(k)$ be the maximum number of leaves of the search tree if the parameter is at most k (let $T(k) = 1$ for $k \leq 0$).

$$T(k) \leq T(k-1) + T(k-3)$$

There is a standard technique for bounding such functions asymptotically.

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There is a standard technique for bounding such functions asymptotically.

We prove by induction that $T(k) \leq c^k$ for some $c > 1$ as small as possible.

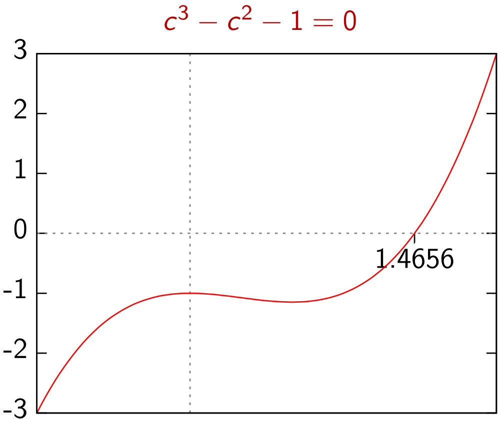
What values of c are good? We need:

$$c^k \geq c^{k-1} + c^{k-3}$$
$$c^3 - c^2 - 1 \geq 0$$

We need to find the roots of the **characteristic equation** $c^3 - c^2 - 1 = 0$.

Note: it is always true that such an equation has a unique positive root.

Better analysis



$c = 1.4656$ is a good value $\Rightarrow$ $T(k) \leq 1.4656^k$
$\Rightarrow$ We have a $O^*(1.4656^k)$ algorithm for Vertex Cover.

Better analysis

We showed that if $T(k) \leq T(k-1) + T(k-3)$, then $T(k) \leq 1.4656^k$ holds.

Is this bound tight? There are two questions:

- Can the function $T(k)$ be that large?
  Yes (ignoring rounding problems).
- Can the search tree of the VERTEX COVER algorithm be that large?
  Difficult question, hard to answer in general.

Branching vectors

The **branching vector** of our $O^*(1.4656^k)$ Vertex Cover algorithm was $(1, 3)$.

Example: Let us bound the search tree for the branching vector $(2, 5, 6, 6, 7, 7)$.
(2 out of the 6 branches decrease the parameter by 7, etc.).

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(2 out of the 6 branches decrease the parameter by 7, etc.).

The value $c > 1$ has to satisfy:

$$c^k \geq c^{k-2} + c^{k-5} + 2c^{k-6} + 2c^{k-7}$$
$$c^7 - c^5 - c^2 - 2c - 2 \geq 0$$

Unique positive root of the characteristic equation: $1.4483 \Rightarrow T(k) \leq 1.4483^k$.

It is hard to compare branching vectors intuitively.

Branching vectors

Example: The roots for branching vector (i, j) $(1 \leq i, j \leq 6)$.

$$T(k) \leq T(k-i) + T(k-j) \Rightarrow c^k \geq c^{k-i} + c^{k-j}$$
$$c^j - c^{j-i} - 1 \geq 0$$

We compute the unique positive root.

	1	2	3	4	5	6
1	2.0000	1.6181	1.4656	1.3803	1.3248	1.2852
2	1.6181	1.4143	1.3248	1.2721	1.2366	1.2107
3	1.4656	1.3248	1.2560	1.2208	1.1939	1.1740
4	1.3803	1.2721	1.2208	1.1893	1.1674	1.1510
5	1.3248	1.2366	1.1939	1.1674	1.1487	1.1348
6	1.2852	1.2107	1.1740	1.1510	1.1348	1.1225

Example: Triangle Free Deletion

Triangle Free Deletion
Given (G, k), remove at most k vertices to make the graph triangle free.

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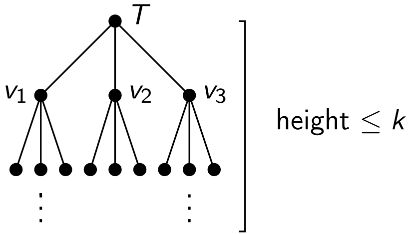

The search tree has at most 3^k leaves and the work to be done is polynomial at each step $\Rightarrow O^*(3^k)$ time algorithm.

Note: If the answer is "NO", then the search tree has **exactly** 3^k leaves.

Graph modification problems

A general problem family containing tasks of the following type:

Given (G, k), do at most k allowed operations on G to make it have property $\mathcal{P}$.

- Allowed operations: vertex deletion, edge deletion, edge addition, $\dots$
- Property $\mathcal{P}$: edgeless, no triangles, no cycles, planar, chordal, regular, disconnected, $\dots$

Examples:

- Vertex Cover: Delete k vertices to make G edgeless.
- Triangle Free Deletion: Delete k vertices to make G triangle free.
- Feedback Vertex Set: Delete k vertices to make G acyclic (forest).

Hereditary properties

Definition

A graph property $\mathcal{P}$ is **hereditary** or **closed under induced subgraphs** if whenever $G \in P$, every induced subgraph of G is also in $\mathcal{P}$.

"removing a vertex does not ruin the property"
(e.g., triangle free, maximum degree ≤ 10, bipartite, planar)

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Observation

Every hereditary property $\mathcal{P}$ can be characterized by a (finite or infinite) set $\mathcal{F}$ of "minimal bad graphs" or "forbidden induced subgraphs": $G \in \mathcal{P}$ if and only if G **does not** have an induced subgraph isomorphic to a member of $\mathcal{F}$.

Example: a graph is bipartite if and only if it does not contain an odd cycle as an induced subgraph.

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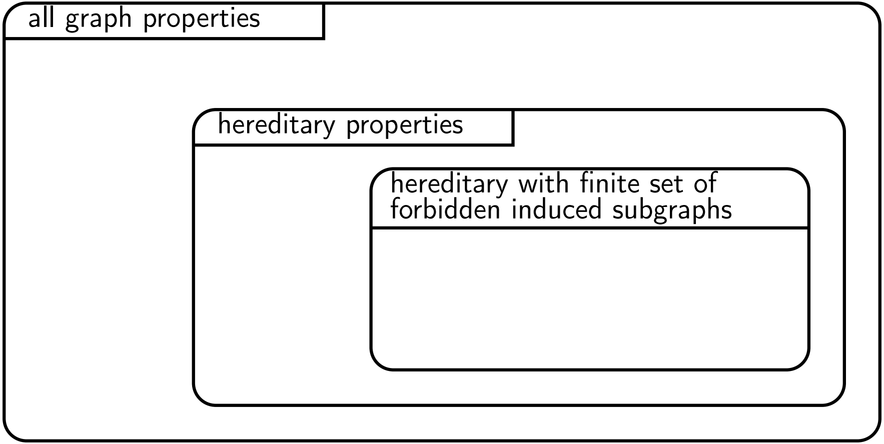
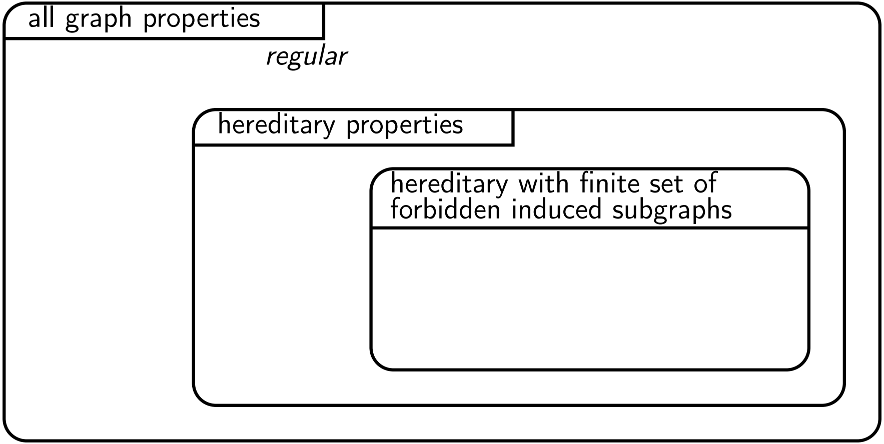
Graph properties



regular *bipartite* *triangle free* *connected*
planar *empty* *complete* *acyclic*

Graph properties



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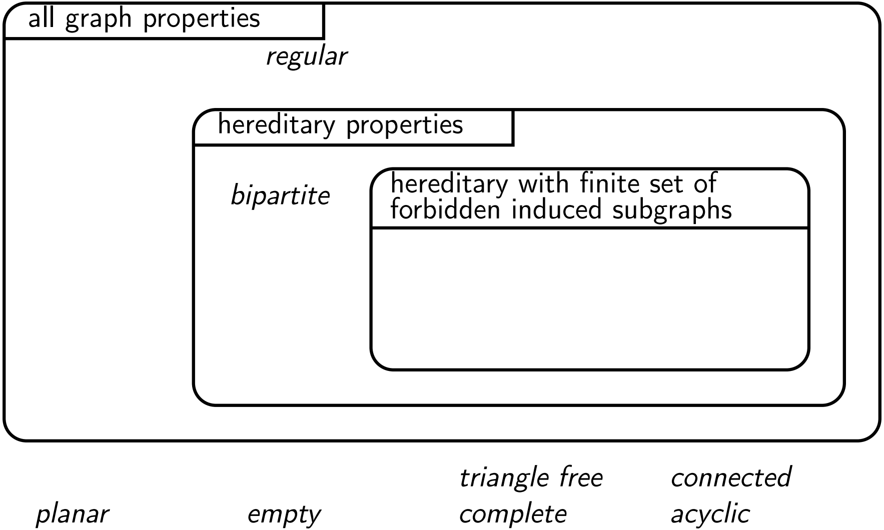

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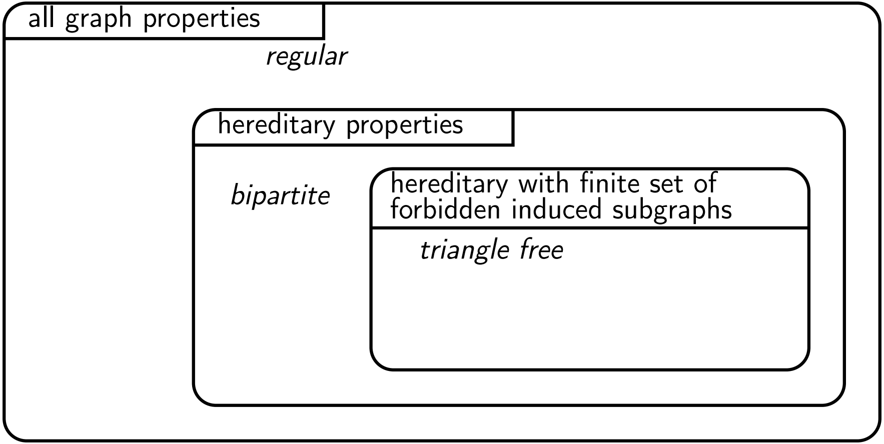

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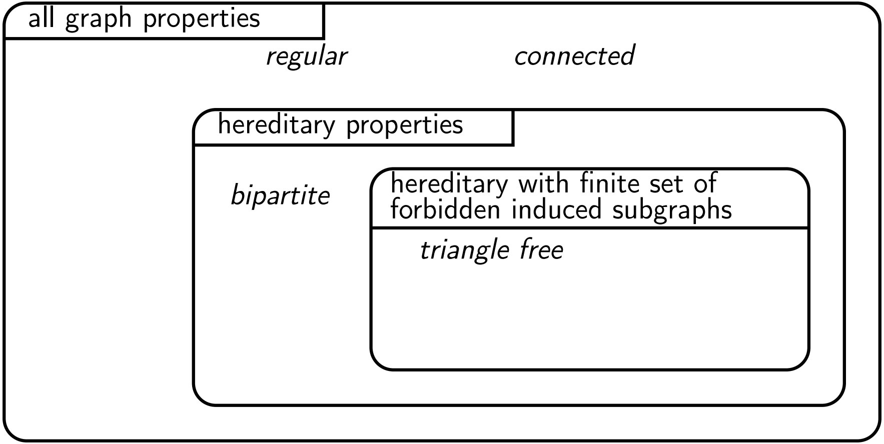

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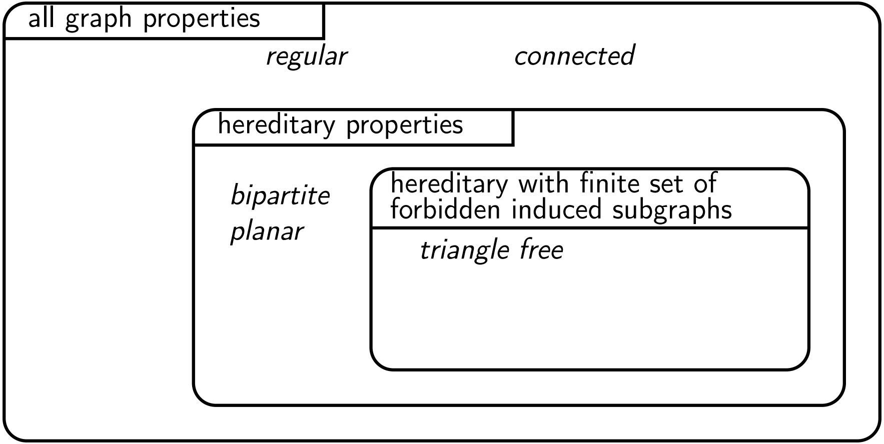

empty *complete* *acyclic*

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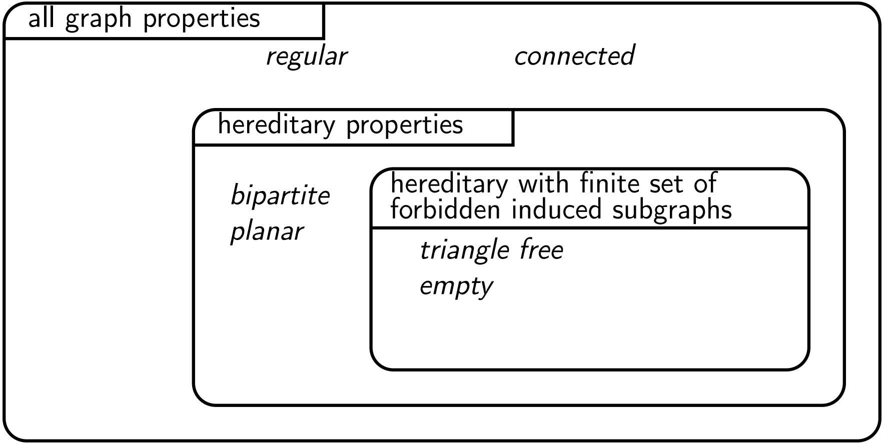
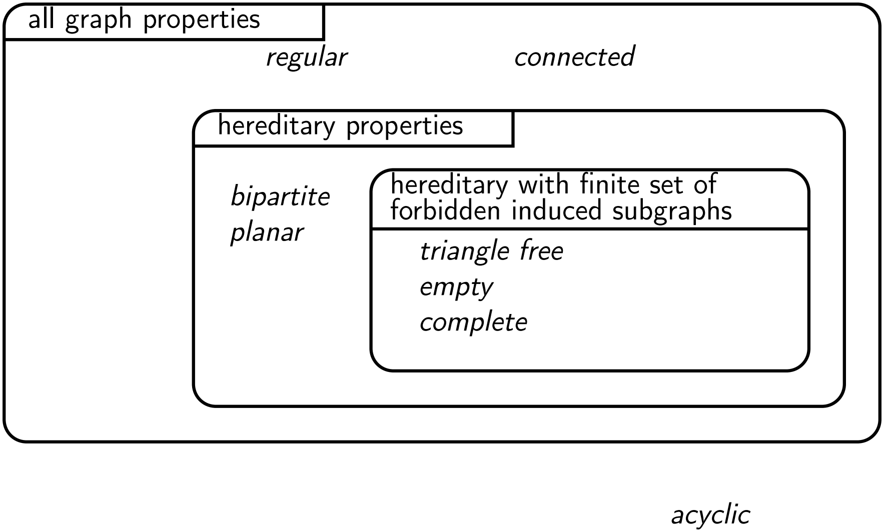


complete *acyclic*

Graph properties



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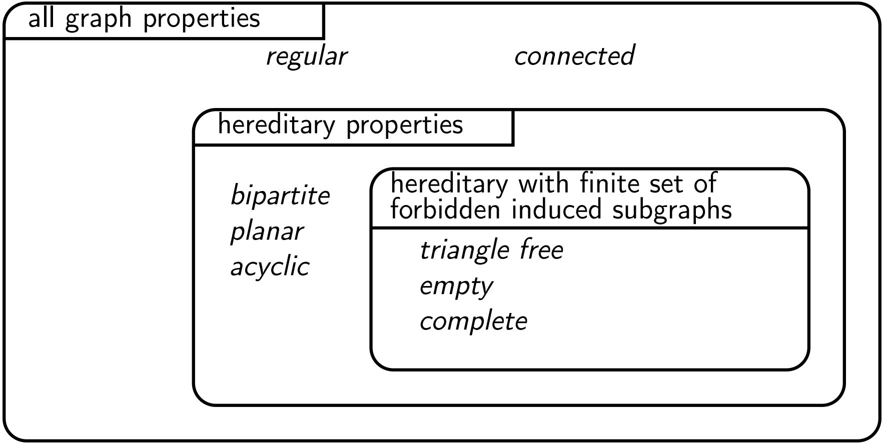

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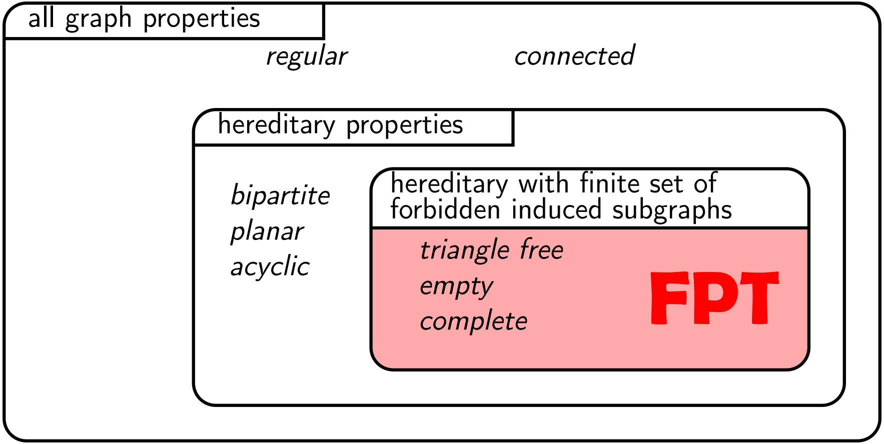


Using finite obstructions

Theorem

If $\mathcal{P}$ is hereditary and can be characterized by a **finite** set $\mathcal{F}$ of forbidden induced subgraphs, then the graph modification problems corresponding to $\mathcal{P}$ are FPT.

Proof:

- Suppose that every graph in $\mathcal{F}$ has at most r vertices. Using brute force, we can find in time $O(n^r)$ a forbidden subgraph (if exists).
- If a forbidden subgraph exists, then we have to delete one of the at most r vertices or add/delete one of the at most $\binom{r}{2}$ edges
  $\Rightarrow$ Branching factor is a constant c depending on $\mathcal{F}$.
- The search tree has at most c^k leaves and the work to be done at each node is $O(n^r)$.

Graph modification problems

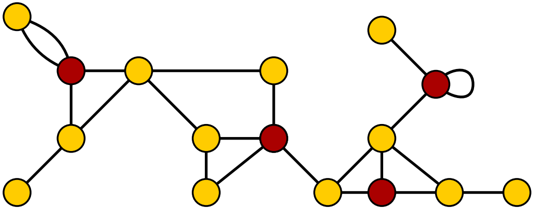
A very wide and active research area in parameterized algorithms.

- If the set of forbidden subgraphs is finite, then the problem is immediately FPT (e.g., Vertex Cover, Triangle Free Deletion). Here the challange is improving the naive running time.
- If the set of forbidden subgraphs is infinite, then very different techniques are needed to show that the problem is FPT (e.g., Feedback Vertex Set, Bipartite Deletion, Planar Deletion).

Feedback Vertex Set

Feedback Vertex Set:
Given (G, k), find a set S of at most k vertices such that $G - S$ has no cycles.

- We allow multiple parallel edges and self loops.
- A **feedback vertex set** is a set that hits every cycle in the graph.



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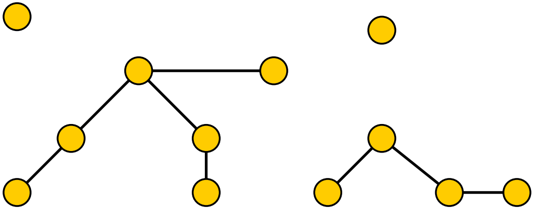


Feedback Vertex Set

- If we find a cycle, then we have to include at least one of its vertices into the solution. But the length of the cycle can be arbitrary large!
- **Main idea:** We identify a set of $O(k)$ vertices such that any size-k feedback vertex set has to contain one of these vertices.
- But first: some reductions to simplify the problem.

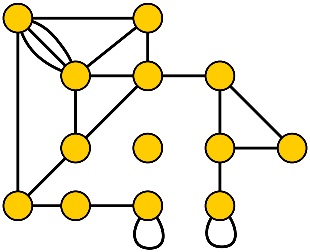
Reduction rules

(R1) If there is a loop at v, then delete v and decrease k by one.

(R2) If there is an edge of multiplicity larger than 2, then reduce its multiplicity to 2.

(R3) If there is a vertex v of degree at most 1, then delete v.

(R4) If there is a vertex v of degree 2, then delete v and add an edge between the neighbors of v.



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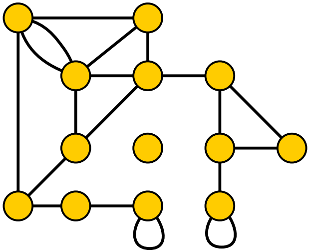

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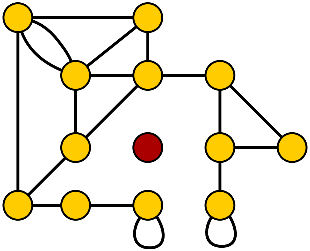

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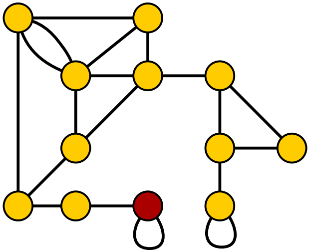

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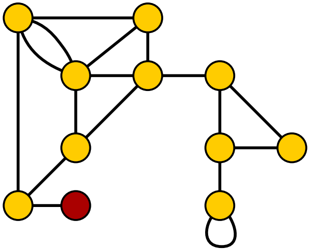

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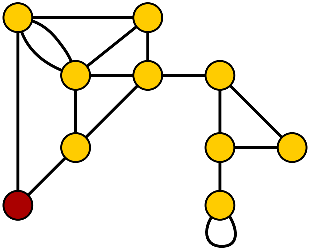

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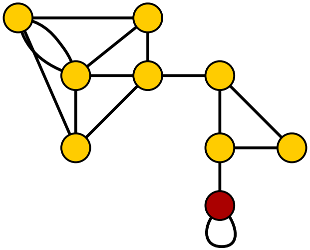

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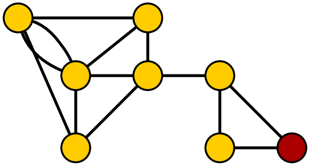

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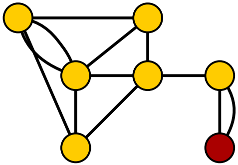

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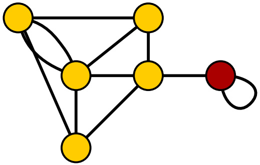

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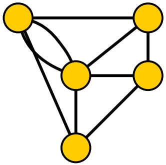

If the reduction rules cannot be applied, then every vertex has degree at least 3.

Branching

Let G be a graph whose vertices have degree at least 3.

- Order the vertices as v_1, v_2, ..., v_n by **decreasing** degree (breaking ties arbitrarily).
- Let $V_{3k} = \{v_1, \ldots, v_{3k}\}$ be the $3k$ largest-degree vertices.

Lemma

If G has minimum degree at least 3, then every feedback vertex set S of size at most k contains a vertex from V_{3k}.

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Algorithm:

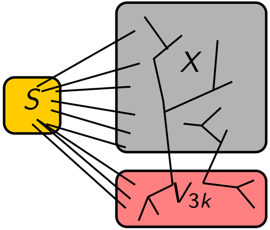
- Apply the reduction rules (poly time) $\Rightarrow$ graph has minimum degree 3.
- For each vertex $v \in V_{3k}$, recurse on the instance $(G - v, k - 1)$.
- Running time $(3k)^k \cdot n^{O(1)} = 2^{O(k \log k)} \cdot n^{O(1)}$.

Proof of the lemma

Lemma

If G has minimum degree at least 3, then every feedback vertex set S of size at most k contains a vertex from V_{3k}.

- $d :=$ minimum degree in V_{3k} ,
  $X = V(G) - (S \cup V_{3k})$.
- Total degree of $V_{3k} \cup X$: $\geq 3kd + 3|X|$
- Edges of $G[V_{3k} \cup X]$: $\leq 3k + |X| - 1$
- Total degree of these edges: $\leq 6k + 2|X| - 2$



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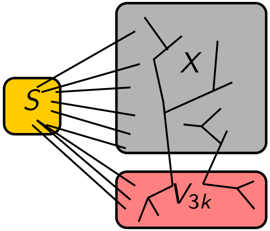
- Edges between S and $V_{3k} \cup X$:
  - $\leq dk$
  - $\geq 3kd + 3|X| - (6k + 2|X| - 2) > 3(d-2)k$
- As $d \geq 3$, we have $3(d-2) \geq d$, contradiction.



Branching: wrap up

- Branching into c directions: $O^*(c^k)$ algorithms.
- Branching into k directions: $O^*(k^k)$ algorithms.
- Branching vectors and analysis of recurrences of the form

$$T(k) = T(k-1) + 2T(k-2) + T(k-3)$$

- Graph modification problems where the graph property can be characterized by a finite set of forbidden induced subgraphs is FPT.

The race for better FPT algorithms