Treewidth: Vol. 1

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Lecture #11
January 18, 2022

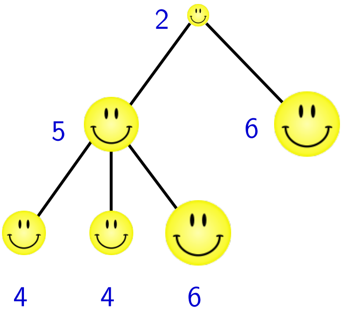
Treewidth

- Treewidth: a notion of “treelike” graphs.
- Some combinatorial properties.
- Algorithmic results.
  - Algorithms on graphs of bounded treewidth.
  - Applications for other problems.

The Party Problem

Party Problem

Problem:	Invite some colleagues for a party.
Maximize:	The total fun factor of the invited people.
Constraint:	Everyone should be having fun.



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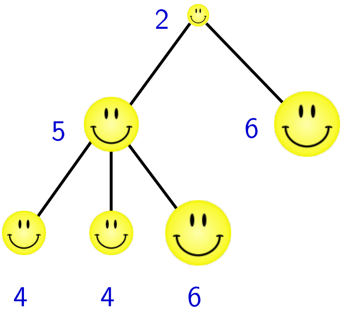

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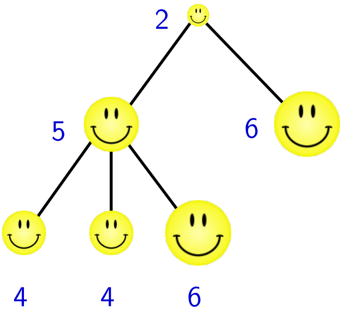

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- **Task:** Find an independent set of maximum weight.

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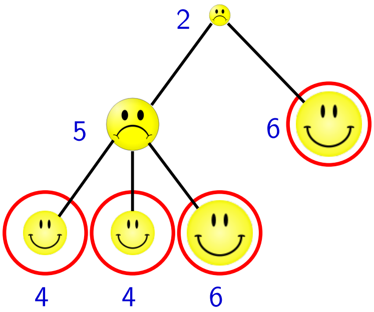

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Solving the Party Problem

Dynamic programming paradigm:
We solve a large number of subproblems that depend on each other. The answer is a single subproblem.

Subproblems:

T_v: the subtree rooted at v.
$A[v]$: max. weight of an independent set in T_v
$B[v]$: max. weight of an independent set in T_v
that does not contain v

Goal: determine $A[r]$ for the root r.

Solving the Party Problem

Subproblems:

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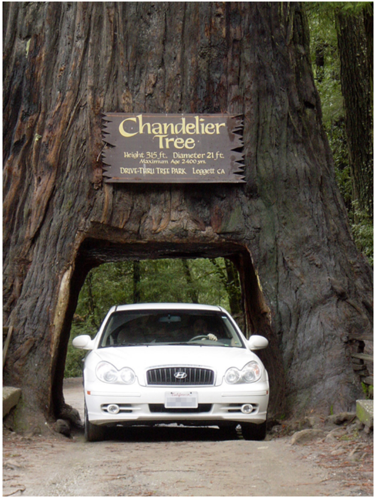
Recurrence:

Assume $v_1, \ldots, v_k$ are the children of v. Use the recurrence relations

$$\boxed{\begin{array}{l} B[v] = \sum_{i=1}^{k} A[v_i] \\ A[v] = \max\{B[v]\ ,\ w(v) + \sum_{i=1}^{k} B[v_i]\} \end{array}}$$

The values $A[v]$ and $B[v]$ can be calculated in a bottom-up order (the leaves are trivial).


Chandelier Tree
Height 315 ft. Diameter 21 ft.
Maximum Age 2400 yrs.
DRIVE-THRU TREE PARK Leggett CA

Treewidth

Generalizing trees

How could we define that a graph is “treelike”?

Generalizing trees

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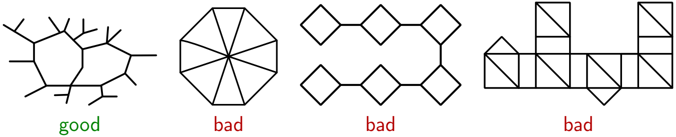
1. Number of cycles is bounded.



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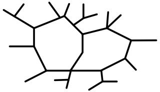
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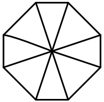
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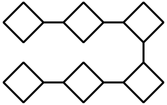
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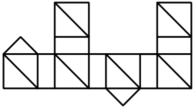
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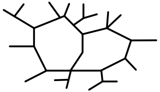
2. Removing a bounded number of vertices makes it acyclic.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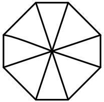
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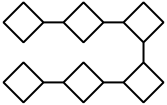
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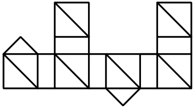
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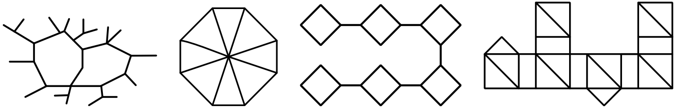

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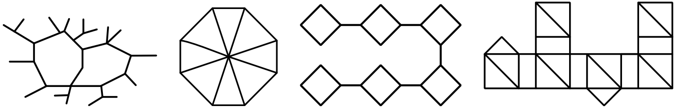
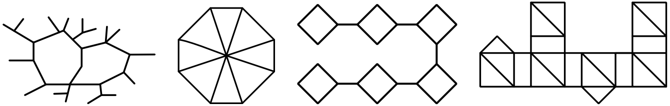


good good bad bad

3. Bounded-size parts connected in a tree-like way.

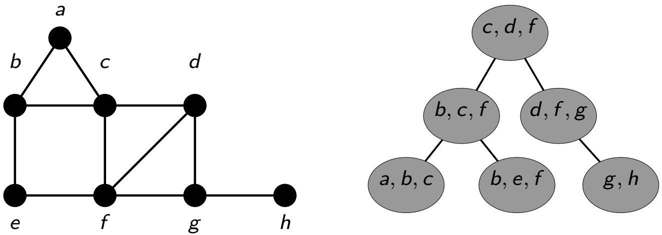


bad bad good good

Treewidth — a measure of "tree-likeness"

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If u and v are neighbors, then there is a bag containing both of them.
2. For every v, the bags containing v form a connected subtree.



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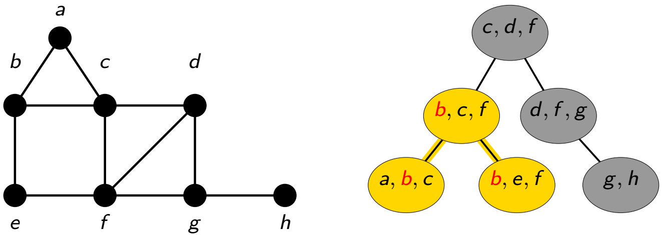

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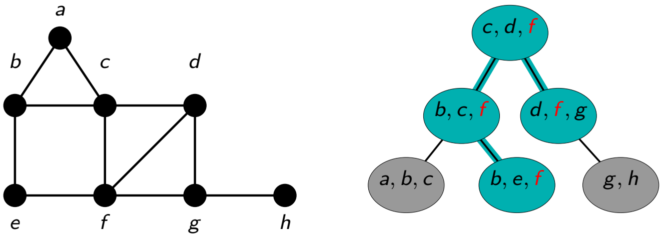

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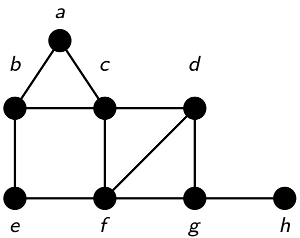
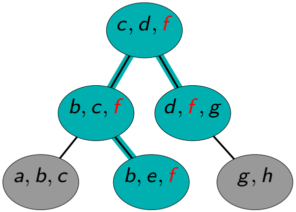
Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.





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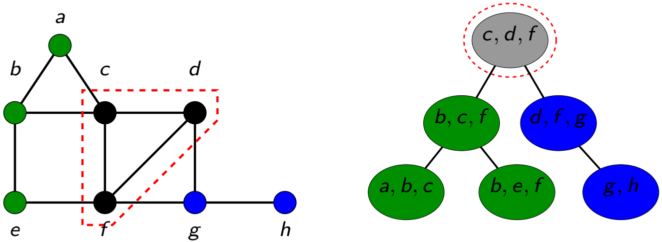

Each bag is a separator.

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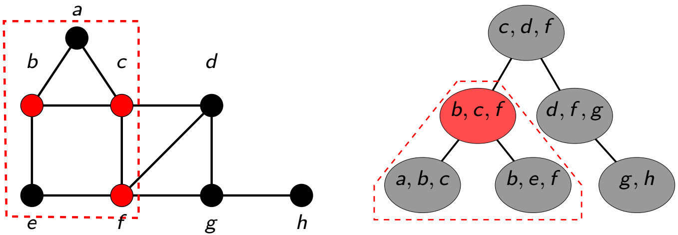
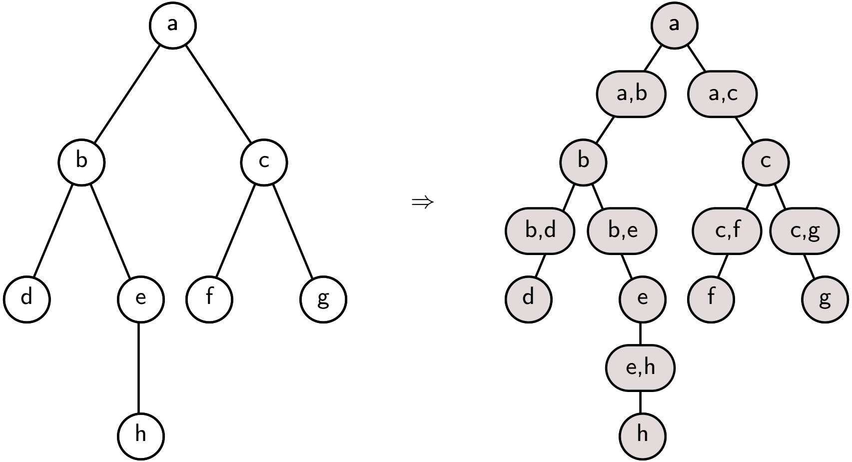


A subtree communicates with the outside world only via the root of the subtree.

Treewidth

Fact: treewidth $= 1 \iff$ graph is a forest



Exercise: A cycle cannot have a tree decomposition of width 1.

Finding tree decompositions

Hardness:

Theorem [Arnborg, Corneil, Proskurowski 1987]

It is NP-hard to determine the treewidth of a graph (given a graph G and an integer w, decide if the treewidth of G is at most w).

Fixed-parameter tractability:

Theorem [Bodlaender 1996]

There is a $2^{O(w^3)} \cdot n$ time algorithm that finds a tree decomposition of width w (if exists).

Consequence:
If we want an FPT algorithm parameterized by treewidth w of the input graph, then we can assume that a tree decomposition of width w is available.

Finding tree decompositions — approximately

Sometimes we can get better dependence on treewidth using approximation.

FPT approximation:

Theorem [Robertson and Seymour]

There is a $O(3^{3w} \cdot w \cdot n^2)$ time algorithm that finds a tree decomposition of width $4w + 1$, if the treewidth of the graph is at most w.

Polynomial-time approximation:

Theorem [Feige, Hajiaghayi, Lee 2008]

There is a polynomial-time algorithm that finds a tree decomposition of width $O(w\sqrt{\log w})$, if the treewidth of the graph is at most w.

Weighted Max Independent Set and treewidth

Theorem

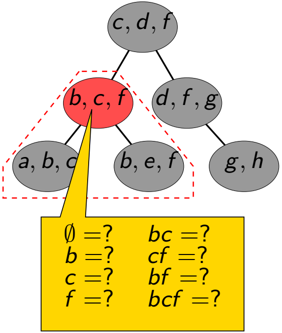
Given a tree decomposition of width w, Weighted Max Independent Set can be solved in time $O(2^w \cdot w^{O(1)} \cdot n)$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

Generalizing our solution for trees:

Instead of computing 2 values $A[v]$, $B[v]$ for each **vertex** of the graph, we compute $2^{|B_x|} \le 2^{w+1}$ values for each bag B_x.

$M[x, S]$:
the max. weight of an independent set $I \subseteq V_x$ with $I \cap B_x = S$.



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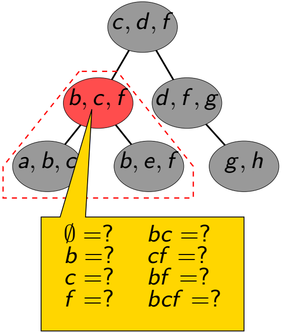

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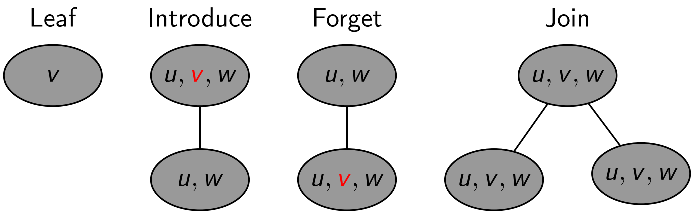
How to determine $M[x, S]$ if all the values are known for the children of x?

Nice tree decompositions

Definition

A rooted tree decomposition is **nice** if every node x is one of the following 4 types:

- **Leaf:** no children, $|B_x| = 1$
- **Introduce:** 1 child y with $B_x = B_y \cup \{v\}$ for some vertex v
- **Forget:** 1 child y with $B_x = B_y \setminus \{v\}$ for some vertex v
- **Join:** 2 children y_1, y_2 with $B_x = B_{y_1} = B_{y_2}$



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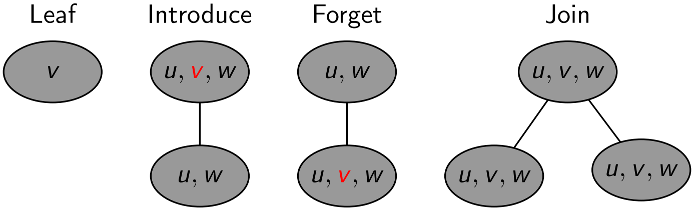
Theorem

A tree decomposition of width w and n nodes can be turned into a nice tree decomposition of width w and $O(wn)$ nodes in time $O(w^2 n)$.

Weighted Max Independent Set
and nice tree decompositions

- **Leaf:** no children, $|B_x| = 1$
  Trivial!
- **Introduce:** 1 child y with $B_x = B_y \cup \{v\}$ for some vertex v

$$M[x,S] = \begin{cases} M[y,S] & \text{if } v \notin S, \\ M[y,S\setminus\{v\}] + w(v) & \text{if } v \in S \text{ but } v \text{ has no neighbor in } S, \\ -\infty & \text{if } S \text{ contains } v \text{ and its neighbor.} \end{cases}$$



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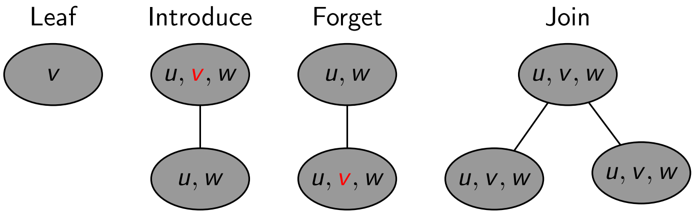

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There are at most $2^{w+1} \cdot n$ subproblems $M[x,S]$ and each subproblem can be solved in $w^{O(1)}$ time
(assuming the children are already solved).

$\Downarrow$

Running time is $O(2^w \cdot w^{O(1)} \cdot n)$.

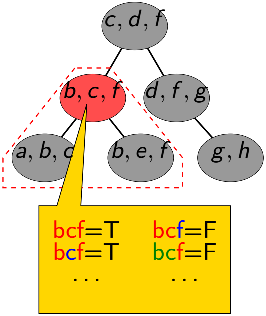
3-Coloring and tree decompositions

Theorem

Given a tree decomposition of width w, 3-Coloring can be solved in $O(3^w \cdot w^{O(1)} \cdot n)$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

For every node x and coloring $c : B_x \to \{1,2,3\}$, we compute the Boolean value $E[x,c]$, which is true if and only if c can be extended to a proper 3-coloring of V_x.



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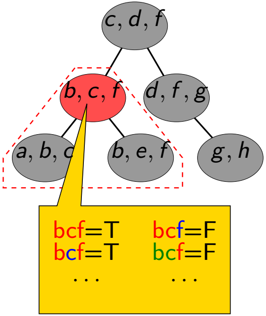

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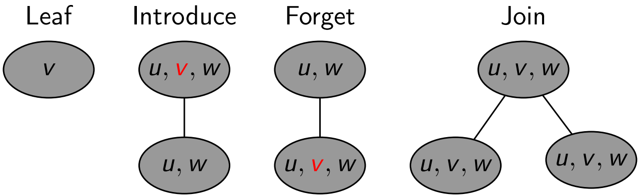

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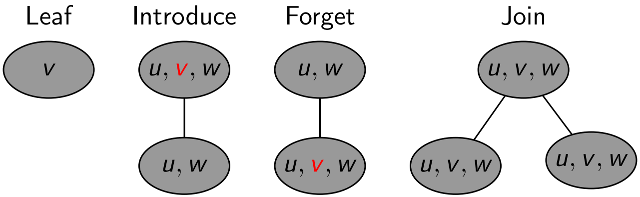

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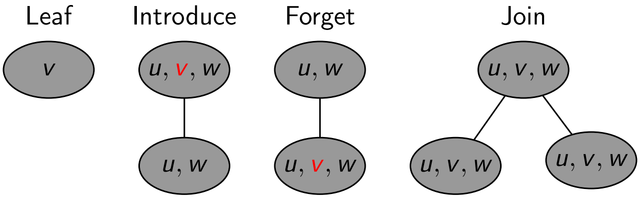

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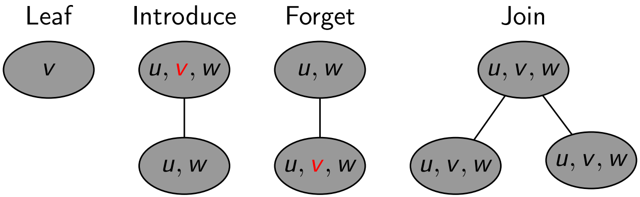

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There are at most $3^{w+1} \cdot n$ subproblems $E[x, c]$ and each subproblem can be solved in $w^{O(1)}$ time (assuming the children are already solved).

$\Rightarrow$ Running time is $O(3^w \cdot w^{O(1)} \cdot n)$.

$\Rightarrow$ 3-Coloring is FPT parameterized by treewidth.

Vertex coloring

More generally:

Theorem

Given a tree decomposition of width w, c-Coloring can be solved in time $c^w \cdot n^{O(1)}$.

Exercise: Every graph of treewidth at most w can be colored with $w + 1$ colors.

Theorem

Given a tree decomposition of width w, Vertex Coloring can be solved in time $O^*(w^w)$.

$\Rightarrow$ Vertex Coloring is FPT parameterized by treewidth.

Dominating Set and treewidth

Dominating Set: Given G and k, find a set S of k vertices such that every vertex of G is in S or has a neighbor in S.

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Dominating Set and treewidth

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- Every vertex in V_x is dominated by D.
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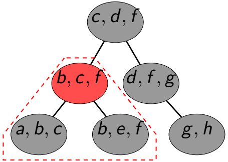

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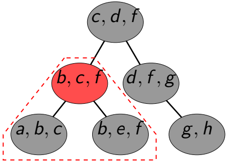

Problem: vertices in B_x can be dominated by vertices outside V_x.

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Second try:

$M[x, S_1, S_2]$: size of the smallest set $D \subseteq V_x$ such that

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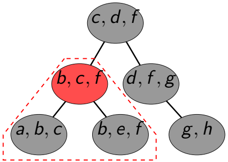
- D dominates every vertex of S_2.
- Every vertex in $V_x \setminus B_x$ is dominated by D.



$\Rightarrow 3^{w+1}$ subproblems at each node x.

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- Renaming "not dominated" to "don't care" can improve to $O(4^w \cdot n)$ time.
- Fast subset convolution: $O(3^w \cdot n)$ time.

Hamiltonian cycle and treewidth

Theorem

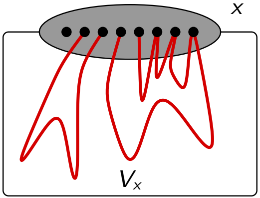
Given a tree decomposition of width w, Hamiltonian cycle can be solved in time $w^{O(w)} \cdot n$.

B_x: vertices appearing in node x.

V_x: vertices appearing in the subtree rooted at x.

If H is a Hamiltonian cycle, then the subgraph $H[V_x]$ is a set of paths with endpoints in B_x.

What are the important properties of $H[V_x]$ "seen from outside"?



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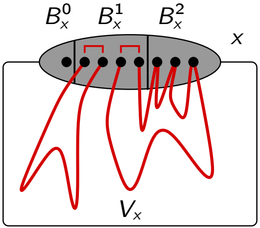


What are the important properties of $H[V_x]$ "seen from outside"?

- The subsets B_x^0, B_x^1, B_x^2 of B_x having degree 0, 1, and 2.
- The matching M of B_x^1.

No. of subproblems (B_x^0, B_x^1, B_x^2, M) for node x: at most $3^w \cdot w^w$.
For each subproblem, we have to determine if there is a set of paths with this pattern.

Other problems

There are other problems where the natural DP needs to keep track of $w^{O(w)}$ possibilties of a partition.

Theorem

Given a tree decomposition of width w, there are $w^{O(w)} \cdot n$ time algorithms for

- Hamiltonian cycle
- Steiner Tree
- Cycle Packing
- ...

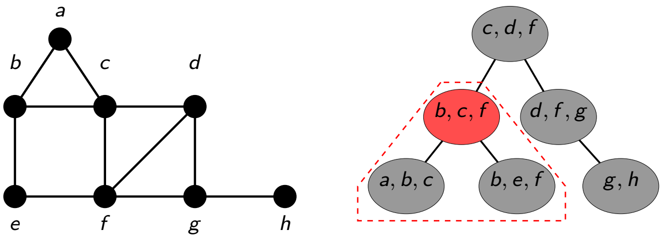
Treewidth — a measure of "tree-likeness"

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If u and v are neighbors, then there is a bag containing both of them.
2. For every v, the bags containing v form a connected subtree.

Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.



Monadic Second Order Logic

Extended Monadic Second Order Logic (EMSO)

A logical language on graphs consisting of the following:

- Logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $=$, $\neq$
- quantifiers $\forall$, $\exists$ over vertex/edge variables
- predicate $\mathrm{adj}(u, v)$: vertices u and v are adjacent
- predicate $\mathrm{inc}(e, v)$: edge e is incident to vertex v
- quantifiers $\forall$, $\exists$ over vertex/edge set variables
- $\in$, $\subseteq$ for vertex/edge sets

Example:
The formula

$$\exists C \subseteq V\,(\exists v \in V \wedge \forall v \in C\ \exists u_1, u_2 \in C(u_1 \neq u_2 \wedge \mathrm{adj}(u_1, v) \wedge \mathrm{adj}(u_2, v)))$$

is true on graph G if and only if ...

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is true on graph G if and only if G has a cycle.

Courcelle's Theorem

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If a graph property can be expressed in EMSO, then for every fixed $w \geq 1$, there is a linear-time algorithm for testing this property on graphs having treewidth at most w.

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There exists an algorithm that, given a width-w tree decomposition of an n-vertex graph G and an EMSO formula ϕ, decides whether G satisfies ϕ in time $f(w, |\phi|) \cdot n$.

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There exists an algorithm that, given a width-w tree decomposition of an n-vertex graph G and an EMSO formula ϕ, decides whether G satisfies ϕ in time $f(w, |\phi|) \cdot n$.

If we can express a property in EMSO, then we immediately get that testing this property is FPT parameterized by the treewidth w of the input graph.

Note: The constant depending on w can be very large (double, triple exponential etc.), therefore a direct dynamic programming algorithm can be more efficient.

Using Courcelle's Theorem

Can we express 3-COLORING and HAMILTONIAN CYCLE in EMSO?

Using Courcelle's Theorem

Can we express 3-Coloring and Hamiltonian Cycle in EMSO?

3-Coloring

$\exists C_1, C_2, C_3 \subseteq V \left(\forall v \in V \,(v \in C_1 \vee v \in C_2 \vee v \in C_3)\right) \wedge \left(\forall u, v \in V \operatorname{adj}(u, v) \to (\neg(u \in C_1 \wedge v \in C_1) \wedge \neg(u \in C_2 \wedge v \in C_2) \wedge \neg(u \in C_3 \wedge v \in C_3))\right)$

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3-Coloring

$\exists C_1, C_2, C_3 \subseteq V\, \big(\forall v \in V\, (v \in C_1 \lor v \in C_2 \lor v \in C_3)\big) \land \big(\forall u, v \in V\, \mathrm{adj}(u, v) \to (\neg(u \in C_1 \land v \in C_1) \land \neg(u \in C_2 \land v \in C_2) \land \neg(u \in C_3 \land v \in C_3))\big)$

Hamiltonian Cycle

$\exists H \subseteq E\big(\text{spanning-connected}(H) \land (\forall v \in V\, \text{degree2}(H, v))\big)$

$\text{degree2}(H, v) := \exists e_1, e_2 \in H\big((e_1 \neq e_2) \land \mathrm{inc}(e_1, v) \land \mathrm{inc}(e_2, v) \land (\forall e_3 \in H \mathrm{inc}(e_3, v) \to (e_1 = e_3 \lor e_2 = e_3))\big)$

$\text{spanning-connected}(H) := \forall Z \subseteq V\big(((\exists v \in V : v \in Z) \land (\exists v \in V : v \notin Z)) \to (\exists e \in H \exists x \in V \exists y \in V : (x \in Z) \land (y \notin Z) \land \mathrm{inc}(e, x) \land \mathrm{inc}(e, y))\big)$

Using Courcelle's Theorem

Three ways of using Courcelle's Theorem:

1. The problem can be described by a single formula (e.g, 3-Coloring, Hamiltonian Cycle).

   $\Rightarrow$ Problem can be solved in time $f(w) \cdot n$ for graphs of treewidth at most w, i.e., FPT parameterized by treewidth.

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2. The problem can be described by a formula for each value of the parameter k.

   Example: For each k, having a cycle of length exactly k can be expressed as

   $$\exists v_1, \ldots, v_k \in V \; ((v_1 \neq v_2) \wedge (v_1 \neq v_3) \wedge \ldots (v_{k-1} \neq v_k))$$
   $$\wedge(\text{adj}(v_1, v_2) \wedge \text{adj}(v_2, v_3) \wedge \cdots \wedge \text{adj}(v_{k-1}, v_k) \wedge \text{adj}(v_k, v_1)).$$

   $\Rightarrow$ Problem can be solved in time $f(k, w) \cdot n$ for graphs of treewidth w, i.e., FPT parameterized with combined parameter k and treewidth w.

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3. Optimization version: find largest set X such that...

Subgraph Isomorphism

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Input: graphs H and G

Find: a subgraph of G isomorphic to H.

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For each H, we can construct a formula ϕ_H that expresses "G has a subgraph isomorphic to H" (similarly to the k-cycle on the previous slide).

$\Rightarrow$ By Courcelle's Theorem, Subgraph Isomorphism can be solved in time $f(H, w) \cdot n$ if G has treewidth at most w.

Subgraph Isomorphism

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Input: graphs H and G
Find: a subgraph of G isomorphic to H.

As there is only a finite number of simple k-vertex graphs, Subgraph Isomorphism can be solved in time $f(k, w) \cdot n$ if H has k vertices and G has treewidth at most w.

Theorem

Subgraph Isomorphism is FPT parameterized by combined parameter $k := |V(H)|$ and the treewidth w of G.

MSO on words

Theorem [Büchi, Elgot, Trakhtenbrot 1960]

If a language $L \subseteq \Sigma^*$ can be defined by an MSO formula ϕ using the relation $<$, then L is regular.

Example: a^*bc^* is defined by

$$\exists x : P_b(x) \wedge (\forall y : (y < x) \rightarrow P_a(y)) \wedge (\forall y : (x < y) \rightarrow P_c(y)).$$

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We prove a more general statement for formulas $\phi(w, X_1, \ldots, X_k)$ and words over $\Sigma \cup \{0, 1\}^k$, where X_i is a subset of positions of w.

Induction over the structure of ϕ:

- FSM for $\neg\phi(w)$, given FSM for $\phi(w)$.
- FSM for $\phi_1(w) \wedge \phi_2(w)$, given FSMs for $\phi_1(w)$ and $\phi_2(w)$.
- FSM for $\exists X \phi(w, X)$, given FSM for $\phi(w, X)$.
- etc.

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Proving Courcelle's Theorem:

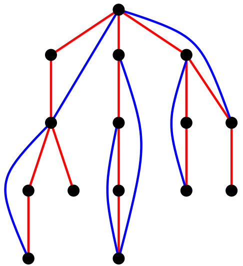
- Generalize from words to trees.
- A width-w tree decomposition can be interpreted as a tree over an alphabet of size $f(w)$.
- Formula $\Rightarrow$ tree automata.

Depth-first search (DFS)

Fact: Finding a cycle of length **at least** k in a graph is FPT parameterized by k.

Let us start a depth-first search from an arbitrary vertex v. There are two types of edges: tree edges and back edges.

- If there is a back edge whose endpoints differ by at least $k-1$ levels $\Rightarrow$ there is a cycle of length at least k.
- Otherwise, the graph has treewidth at most $k-2$ and we can solve the problem by applying Courcelle's Theorem.



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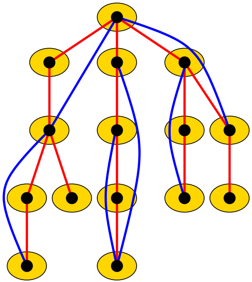

In the second case, a tree decomposition can be easily found: the decomposition has the same structure as the DFS spanning tree and each bag contains the vertex and its $k-2$ ancestors.

Running times

We have seen:

- Independent Set: 2^w
- Vertex Cover: 2^w
- Dominating Set: 3^w
- 3-Coloring: 3^w
- Vertex Coloring: $2^{O(w \log w)}$
- Hamiltonian Cycle: $2^{O(w \log w)}$

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Can we improve on any of these running times?

Hamiltonian Cycle can be improved to $2^{O(w)}$, but lower bounds show that the other algorithms are essentially optimal.

Algorithms — overview

- Algorithms exploit the fact that a subtree communicates with the rest of the graph via a single bag.
- Key point: defining the subproblems.
- Courcelle's Theorem makes this process automatic for many problems.

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