Treewidth: Vol. 3

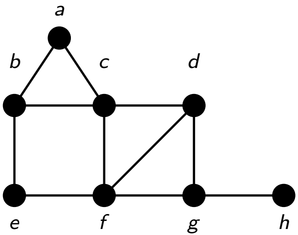
Dániel Marx

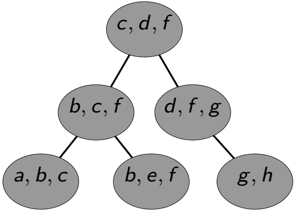
Lecture #13
February 1, 2022

Treewidth — a measure of "tree-likeness"

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If u and v are neighbors, then there is a bag containing both of them.
2. For every v, the bags containing v form a connected subtree.





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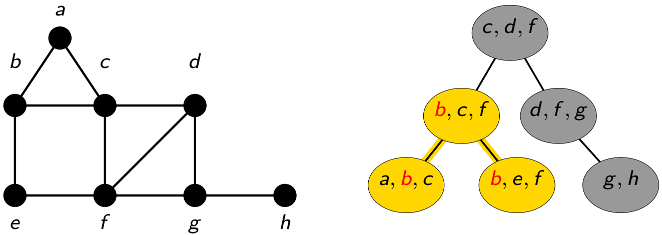

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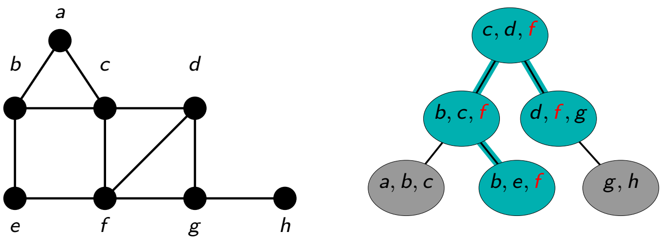

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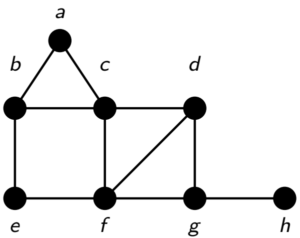
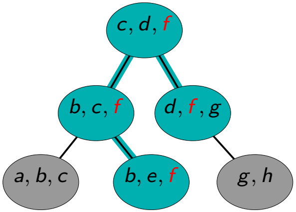
Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.





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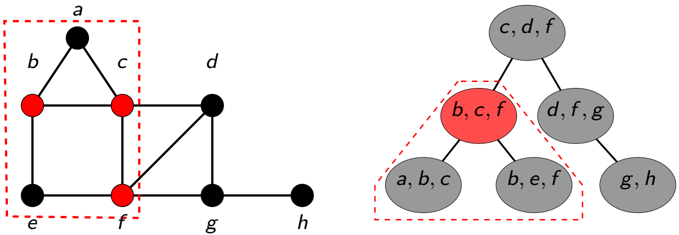


A subtree communicates with the outside world
only via the root of the subtree.

Weighted Max Independent Set and treewidth

Theorem

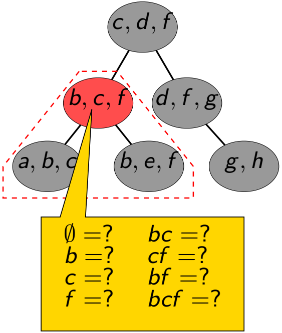
Given a tree decomposition of width w, Weighted Max Independent Set can be solved in time $O(2^w \cdot w^{O(1)} \cdot n)$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

Generalizing our solution for trees:

Instead of computing 2 values $A[v]$, $B[v]$ for each **vertex** of the graph, we compute $2^{|B_x|} \leq 2^{w+1}$ values for each bag B_x.

$M[x, S]$:
the max. weight of an independent set $I \subseteq V_x$ with $I \cap B_x = S$.



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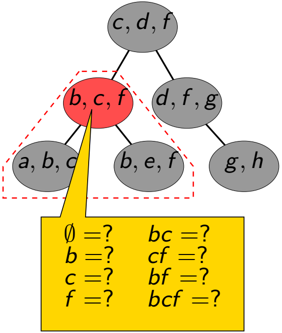

How to determine $M[x, S]$ if all the values are known for the children of x?

Monadic Second Order Logic

Extended Monadic Second Order Logic (EMSO)

A logical language on graphs consisting of the following:

- Logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $=$, $\neq$
- quantifiers $\forall$, $\exists$ over vertex/edge variables
- predicate $\text{adj}(u, v)$: vertices u and v are adjacent
- predicate $\text{inc}(e, v)$: edge e is incident to vertex v
- quantifiers $\forall$, $\exists$ over vertex/edge set variables
- $\in$, $\subseteq$ for vertex/edge sets

Example:
The formula

$$\exists C \subseteq V \forall v \in C \; \exists u_1, u_2 \in C(u_1 \neq u_2 \wedge \text{adj}(u_1, v) \wedge \text{adj}(u_2, v))$$

is true on graph G if and only if G has a cycle.

Courcelle's Theorem

Courcelle's Theorem

There exists an algorithm that, given a width-w tree decomposition of an n-vertex graph G and an EMSO formula ϕ, decides whether G satisfies ϕ in time $f(w, |\phi|) \cdot n$.

If we can express a property in EMSO, then we immediately get that testing this property is FPT parameterized by the treewidth w of the input graph.

$\Rightarrow$ The following problem are FPT parameterized by treewidth:

- c-Coloring
- Hamiltonian Cycle
- Partition into Triangles
- $\ldots$

Subgraph Isomorphism

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Input: graphs H and G

Find: a subgraph of G isomorphic to H.

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Find: a subgraph of G isomorphic to H.

For each H, we can construct a formula ϕ_H that expresses "G has a subgraph isomorphic to H".

$\Rightarrow$ By Courcelle's Theorem, Subgraph Isomorphism can be solved in time $f(H, w) \cdot n$ if G has treewidth at most w.

Theorem

Subgraph Isomorphism is FPT parameterized by combined parameter $k := |V(H)|$ and the treewidth w of G.

Finding tree decompositions

Fixed-parameter tractability:

Theorem [Bodlaender 1996]

There is a $2^{O(w^3)} \cdot n$ time algorithm that finds a tree decomposition of width w (if exists).

Sometimes we can get better dependence on treewidth using approximation.

FPT approximation:

Theorem

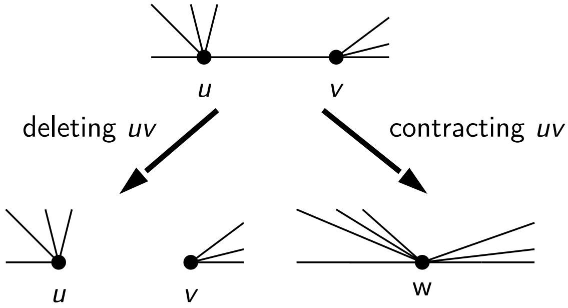
There is a $O(3^{3w} \cdot w \cdot n^2)$ time algorithm that finds a tree decomposition of width $4w + 1$, if the treewidth of the graph is at most w.

Minor

An operation similar to taking subgraphs:

Definition

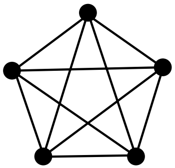
Graph H is a **minor** of G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.



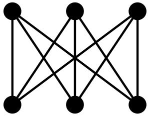
A classical result

Theorem [Kuratowski 1930]

A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$.



K_5



$K_{3,3}$

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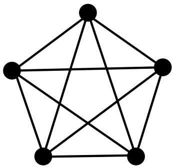
Theorem [Wagner 1937]

A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ as minor.



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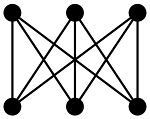


$K_{3,3}$

Graph Minors Theory

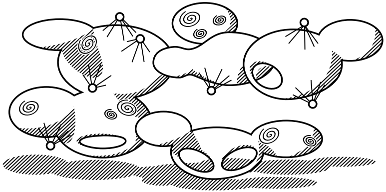
Neil Robertson

Paul Seymour

Theory of graph minors developed in the monumental series

Graph Minors I–XXIII.
J. Combin. Theory, Ser. B
1983–2012

- Structure theory of graphs excluding minors (and much more).
- Galactic combinatorial bounds and running times.
- Important early influence for parameterized algorithms.



Properties of treewidth

Fact: Treewidth does not increase if we delete edges, delete vertices, or contract edges.

$\Rightarrow$ If F is a **minor** of G, then the treewidth of F is at most the treewidth of G.

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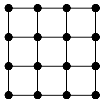
Fact: For every clique K, there is a bag B with $K \subseteq B$.

Fact: The treewidth of the k-clique is $k-1$.

Fact: For every $k \geq 2$, the treewidth of the $k \times k$ grid is exactly k.



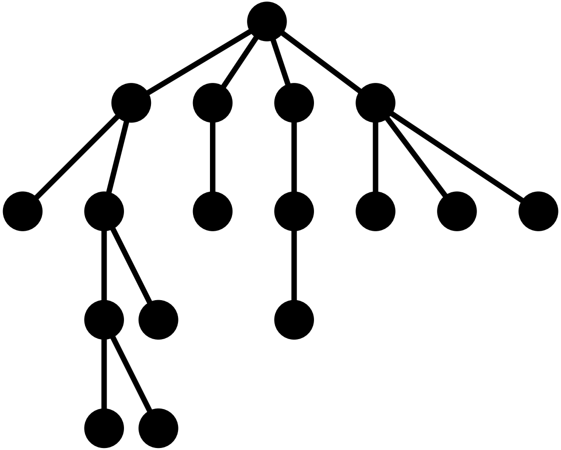
The Cops and Robber game

Game: k cops try to capture a robber in the graph.

- In each step, (a subset of) the cops can move from vertex to vertex arbitrarily with helicopters.
- The robber moves infinitely fast on the edges, cannot move through the cops staying on the ground, and sees where the cops will land.

The Cops and Robber game

Example: 2 cops have a winning strategy in a tree.



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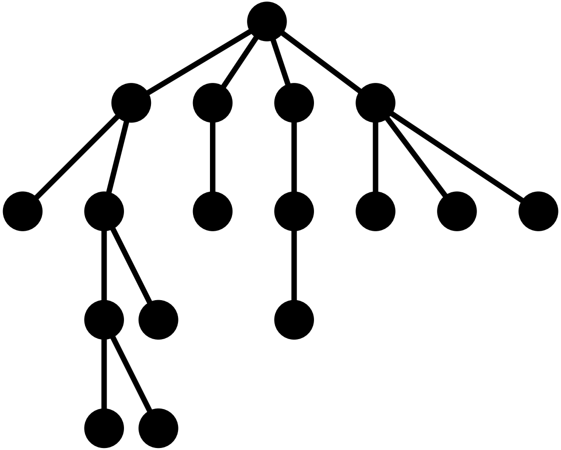

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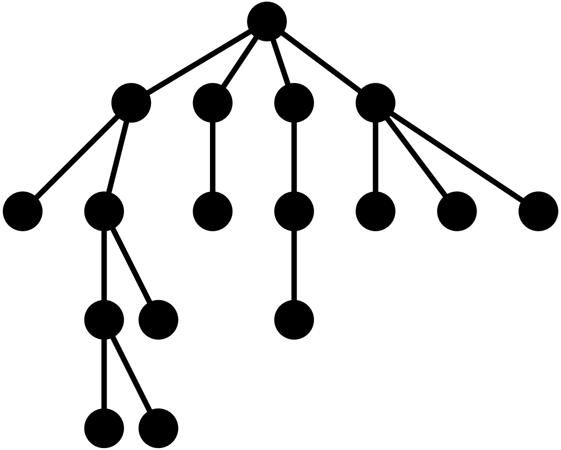

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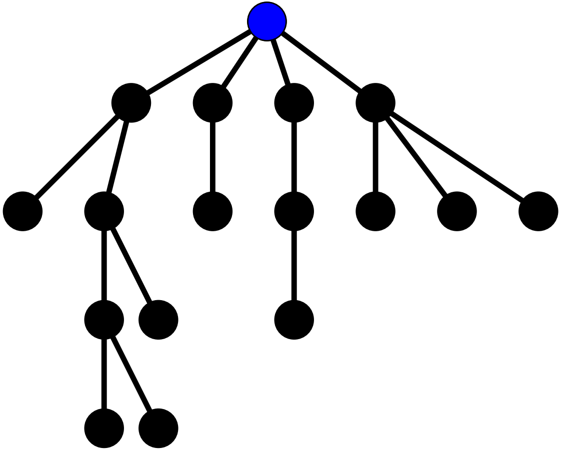

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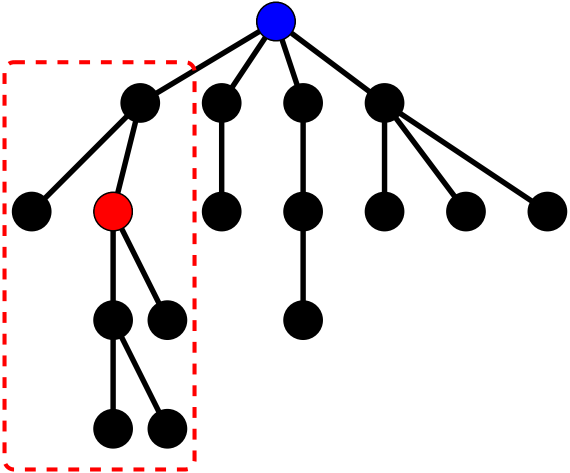

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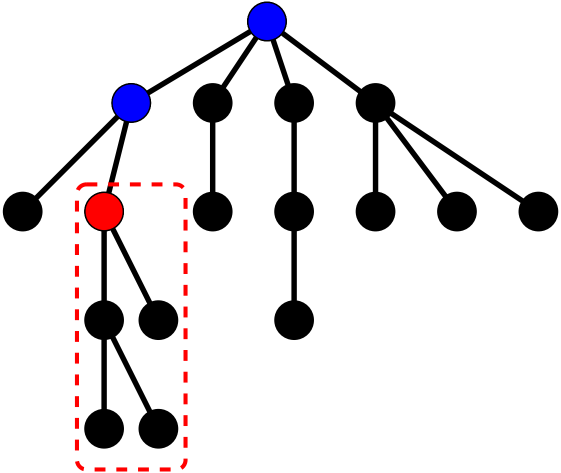

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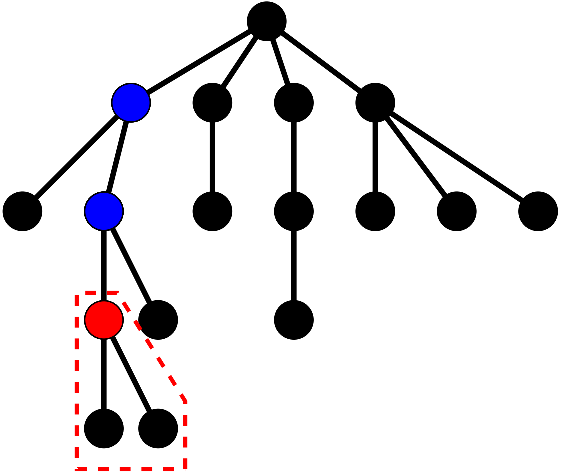

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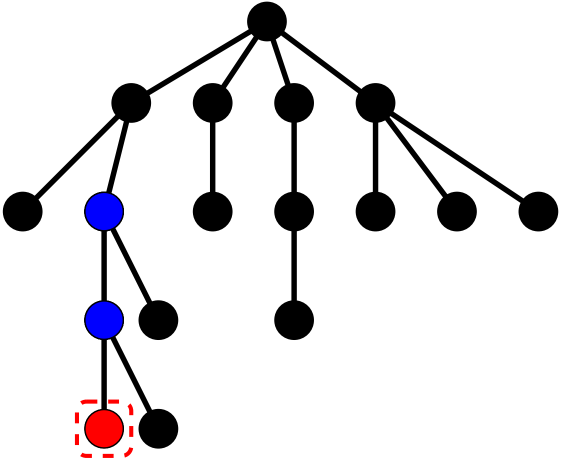

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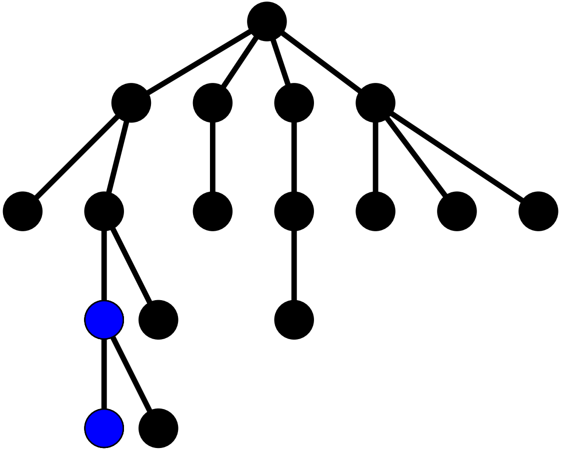

The Cops and Robber game

Theorem [Seymour and Thomas 1993]

$k+1$ cops can win the game $\iff$ the treewidth of the graph is at most k.

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Consequence 1: Algorithms

The winner of the game can be determined in time $n^{O(k)}$ using standard techniques (there are at most n^k positions for the cops)

$$\Downarrow$$

For every fixed k, it can be checked in polynomial time if treewidth is at most k.

(But $f(k) \cdot n^{O(1)}$ algorithms are also known with different techniques!)

The Cops and Robber game

Theorem [Seymour and Thomas 1993]

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Consequence 2: Lower bounds

Exercise 1:
Show that the treewidth of the $k \times k$ grid is at least $k-1$.
(E.g., robber can win against $k-1$ cops.)

Exercise 2:
Show that the treewidth of the $k \times k$ grid is at least k.
(E.g., robber can win against k cops.)

Excluded Grid Theorem

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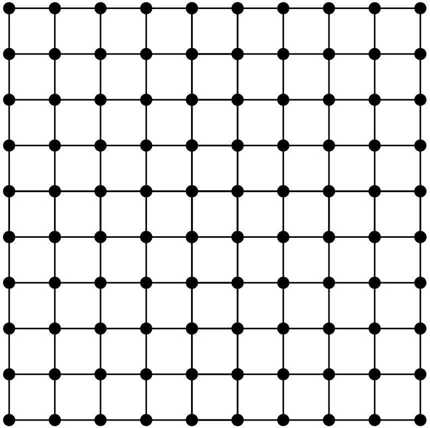
If the treewidth of G is $\Omega(k^9 \log k)$, then G has a $k \times k$ grid minor.



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If the treewidth of G is $\Omega(k^9 \log k)$, then G has a $k \times k$ grid minor.

A large grid minor is a "witness" that treewidth is large, but the relation is approximate:

$k \times k$ grid minor	$\Longrightarrow$	treewidth $\geq k$
tree width $\Omega(k^9 \log k)$	$\Longrightarrow$	$k \times k$ grid minor

Excluded Grid Theorem

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If the treewidth of G is $\Omega(k^9 \log k)$, then G has a $k \times k$ grid minor.

Observation: Every planar graph is the minor of a sufficiently large grid.

Consequence

If H is planar, then every H-minor free graph has treewidth at most $f(H)$.

Planar Excluded Grid Theorem

For planar graphs, we get linear instead of exponential dependence:

Theorem

Every **planar graph** with treewidth at least $5k$ has a $k \times k$ grid minor.

$k \times k$ grid minor	$\Longrightarrow$	treewidth $\geq k$
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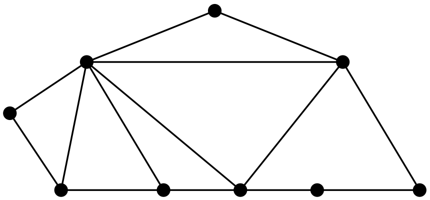
Theorem

An n-vertex planar graph has treewidth $O(\sqrt{n})$.

Outerplanar graphs

Definition

A planar graph is **outerplanar** if it has a planar embedding where every vertex is on the infinite face.



Fact

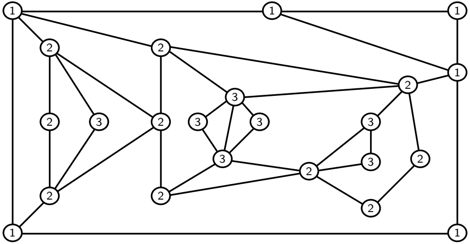
Every outerplanar graph has treewidth at most 2.

k-outerplanar graphs

Given a planar embedding, we can define **layers** by iteratively removing the vertices on the infinite face.

Definition

A planar graph is k-**outerplanar** if it has a planar embedding having at most k layers.



Fact

Every k-outerplanar graph has treewidth at most $3k + 1$.

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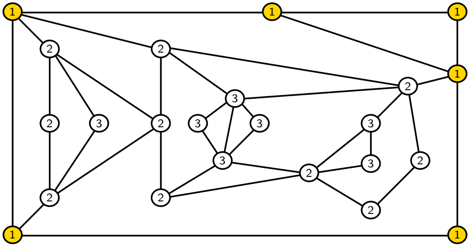

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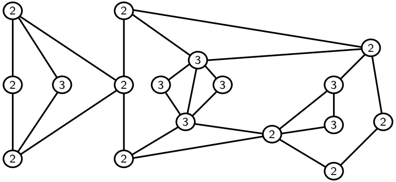

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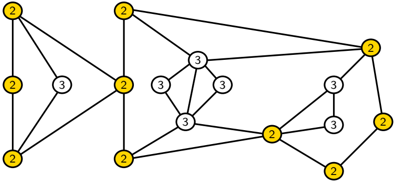

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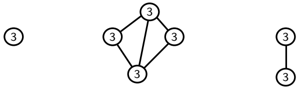

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Treewidth — outline

1. Basic algorithms
2. Complexity results
3. Combinatorial properties
4. Applications
   - The shifting technique
   - Bidimensionality

Approximation schemes

Definition

A **polynomial-time approximation scheme (PTAS)** for a problem P is an algorithm that takes an instance of P and a rational number $\epsilon > 0$,

- always finds a $(1+\epsilon)$-approximate solution,
- the running time is polynomial in n for every fixed $\epsilon > 0$.

Typical running times: $2^{1/\epsilon} \cdot n$, $n^{1/\epsilon}$, $(n/\epsilon)^2$, n^{1/ϵ^2}.

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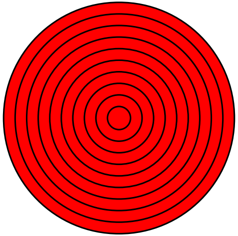
Some classical problems that have a PTAS:

- Independent Set for planar graphs
- TSP in the Euclidean plane
- Steiner Tree in planar graphs
- Knapsack

Baker's shifting strategy for PTAS

Theorem

There is a $2^{O(1/\epsilon)} \cdot n$ time PTAS for Independent Set for planar graphs.



- Let $D := 1/\epsilon$. For a fixed $0 \leq s < D$, delete every layer L_i with $i = s \pmod{D}$

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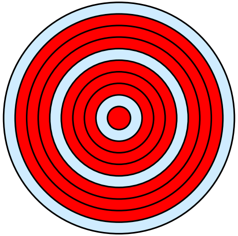

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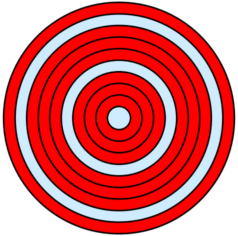

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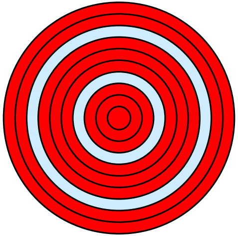

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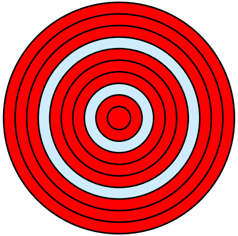

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Baker's shifting strategy for PTAS

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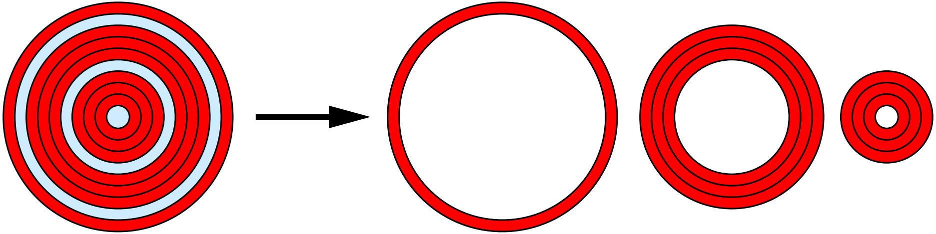
There is a $2^{O(1/\epsilon)} \cdot n$ time PTAS for Independent Set for planar graphs.



- Let $D := 1/\epsilon$. For a fixed $0 \le s < D$, delete every layer L_i with $i = s \pmod D$
- The resulting graph is D-outerplanar, hence it has treewidth at most $3D + 1 = O(1/\epsilon)$.
- Using the $2^{O(\mathrm{tw})} \cdot n$ time algorithm for Independent Set, the problem on the D-outerplanar graph can be solved in time $2^{O(1/\epsilon)} \cdot n$.

Baker's shifting strategy for PTAS

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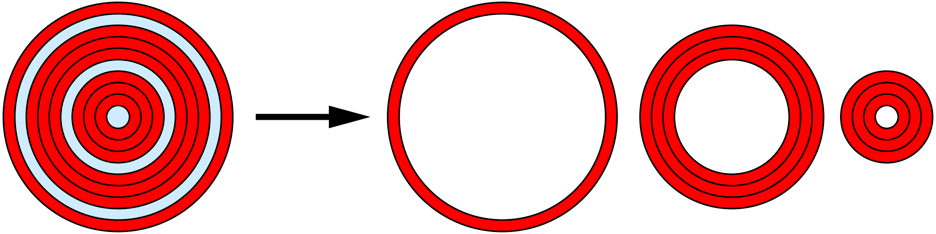
There is a $2^{O(1/\epsilon)} \cdot n$ time PTAS for INDEPENDENT SET for planar graphs.



We do this for every $0 \leq s < D$:
for at least one value of s, we delete
at most $1/D = \epsilon$ fraction of the solution

$\Downarrow$

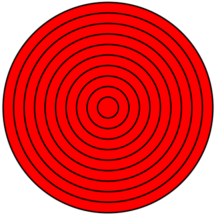
We get a $(1+\epsilon)$-approximate solution.

Baker's shifting strategy for FPT

Subgraph Isomorphism

Input: graphs H and G

Find: a subgraph G isomorphic to H.



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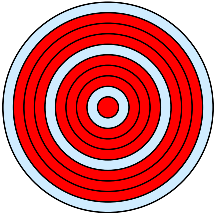

- For a fixed $0 \leq s < k+1$, delete every layer L_i with $i = s \pmod{k+1}$

Baker's shifting strategy for FPT

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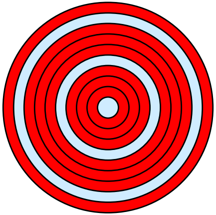

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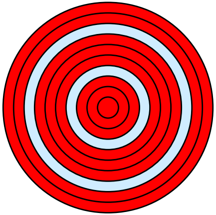

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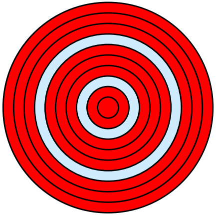

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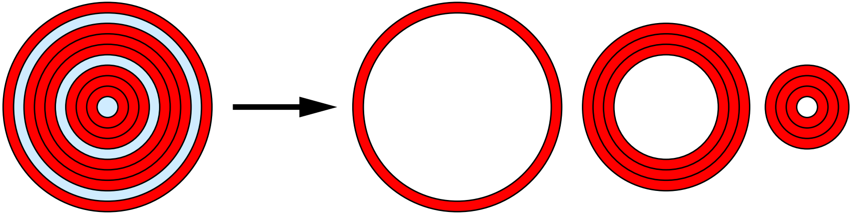

- For a fixed $0 \le s < k+1$, delete every layer L_i with $i = s \pmod{k+1}$
- The resulting graph is k-outerplanar, hence it has treewidth at most $3k+1$.
- Using the $f(k, \mathrm{tw}) \cdot n$ time algorithm for Subgraph Isomorphism, the problem can be solved in time $f(k, 3k+1) \cdot n$.

Baker's shifting strategy for FPT

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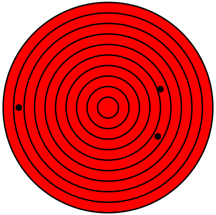

We do this for every $0 \leq s < k+1$:
for at least one value of s, we do not delete
any of the k vertices of the solution

$\Downarrow$

We find a copy of H in G if there is one.

Baker's shifting strategy for FPT

Subgraph Isomorphism

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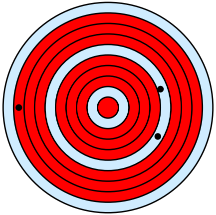

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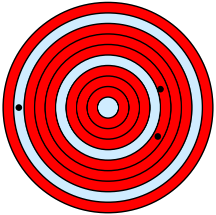

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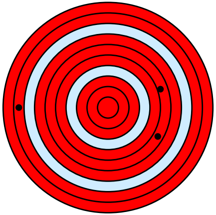

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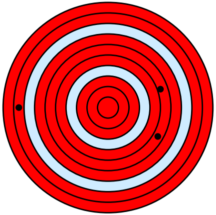


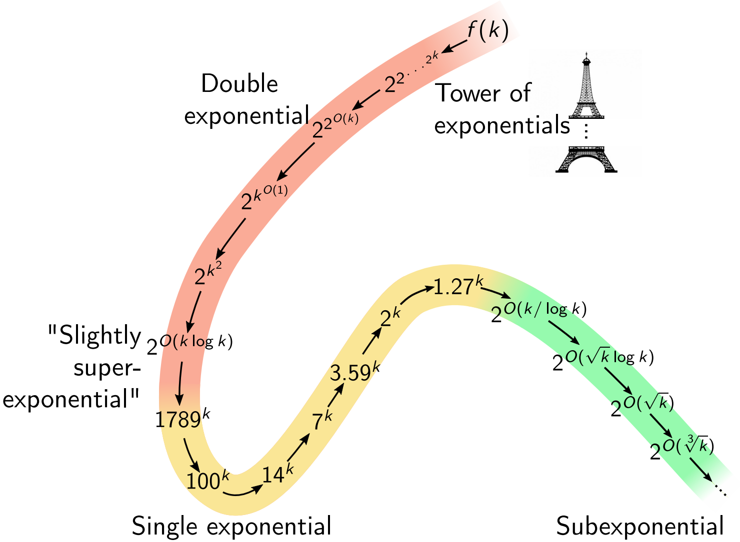
Theorem

SUBGRAPH ISOMORPHISM for planar graphs is FPT parameterized by $k := |V(H)|$.

Baker's shifting strategy for FPT

- The technique is very general, works for many problems on planar graphs:
  - Independent Set
  - Vertex Cover
  - Dominating Set
  - k-Path
  - ...
- More generally: First-Order Logic problems.
- But for some of these problems, much better techniques are known (see the following slides).

The race for better FPT algorithms



Square root phenomenon

Most NP-hard problems (e.g., 3-Coloring, Independent Set, Hamiltonian Cycle, Steiner Tree, etc.) remain NP-hard on planar graphs.[1]

[1] Notable exception: Max Cut is in P for planar graphs.

Square root phenomenon

Most NP-hard problems (e.g., 3-Coloring, Independent Set, Hamiltonian Cycle, Steiner Tree, etc.) remain NP-hard on planar graphs.[1]

The running time is still exponential, but significantly smaller:

$$\begin{aligned} 2^{O(n)} &\Rightarrow 2^{O(\sqrt{n})} \\ n^{O(k)} &\Rightarrow n^{O(\sqrt{k})} \\ 2^{O(k)} \cdot n^{O(1)} &\Rightarrow 2^{O(\sqrt{k})} \cdot n^{O(1)} \end{aligned}$$

Example: A planar n-vertex graph has treewidth $2^{O(\sqrt{n})}$ $\Rightarrow$ 3-Coloring can be solved in time $2^{O(\sqrt{n})}$ in planar graphs.

[1] Notable exception: Max Cut is in P for planar graphs.

Vertex Cover

Theorem

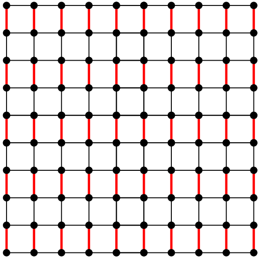
Vertex Cover can be solved in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$ in planar graphs.

We need two facts:

- Removing an edge, removing a vertex, contracting an edge cannot increase the vertex cover number.
- Vertex Cover can be solved in time $2^{w} \cdot n^{O(1)}$ if a tree decomposition of width w is given.

Vertex Cover

Observation: If the treewidth of a planar graph G is at least $5\sqrt{2k}$
$\Rightarrow$ It has a $\sqrt{2k} \times \sqrt{2k}$ grid minor (Planar Excluded Grid Theorem)
$\Rightarrow$ The grid has a matching of size k
$\Rightarrow$ Vertex cover size is at least k in the grid.
$\Rightarrow$ Vertex cover size is at least k in G.



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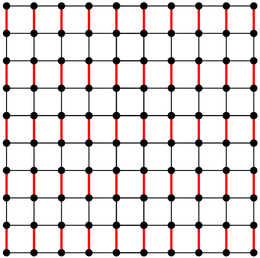
$\Rightarrow$ Vertex cover size is at least k in G.

We use this observation to solve Vertex Cover on planar graphs:

- If treewidth is at least $5\sqrt{2k}$: we answer "vertex cover is $\geq k$."
- If treewidth is less than $5\sqrt{2k}$, then we can solve the problem in time $2^{O(5\sqrt{2k})} \cdot n^{O(1)} = 2^{O(\sqrt{k})} \cdot n^{O(1)}$.



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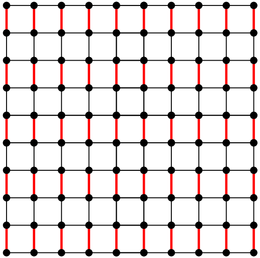
We use this observation to solve Vertex Cover on planar graphs:

- Set $w := 5\sqrt{2k}$.
- Find a 4-approximate tree decomposition.
  - If treewidth is at least w: we answer "vertex cover is $\geq k$."
  - If we get a tree decomposition of width $4w$, then we can solve the problem in time $2^{O(w)} \cdot n^{O(1)} = 2^{O(\sqrt{k})} \cdot n^{O(1)}$.



Bidimensionality

A powerful framework for efficient algorithms on planar graphs.

Setup:

- Let $x(G)$ be some graph invariant (i.e., an integer associated with each graph).
- Given G and k, we want to decide if $x(G) \leq k$ (or $x(G) \geq k$).
- Typical examples:
  - Maximum independent set size.
  - Minimum vertex cover size.
  - Length of the longest path.
  - Minimum dominating set size.
  - Minimum feedback vertex set size.

Bidimensionality

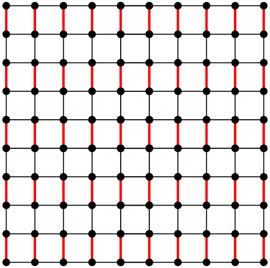
For many natural invariants, we can do this in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$ on planar graphs.

Bidimensionality

Definition

A graph invariant $x(G)$ is **minor-bidimensional** if

- $x(G') \leq x(G)$ for every minor G' of G, and
- If G_k is the $k \times k$ grid, then $x(G_k) \geq ck^2$ (for some constant $c > 0$).



Examples: minimum vertex cover, length of the longest path, feedback vertex set are minor-bidimensional.

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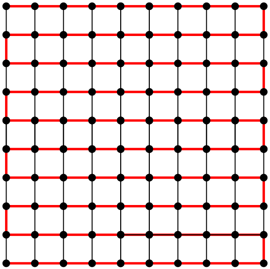

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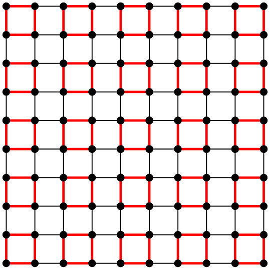

Examples: minimum vertex cover, length of the longest path, feedback vertex set are minor-bidimensional.

Bidimensionality (cont.)

We can answer "$x(G) \geq k$?" for a minor-bidimensional invariant the following way:

- Set $w := c\sqrt{k}$ for an appropriate constant c.
- Use the 4-approximation tree decomposition algorithm.
  - If treewidth is at least w: $x(G)$ is at least k.
  - If we get a tree decomposition of width $4w$, then we can solve the problem using dynamic programming on the tree decomposition.

Running time:

- If we can solve the problem on tree decomposition of width w in time $2^{O(w)} \cdot n^{O(1)}$, then the running time is $2^{O(\sqrt{k})} \cdot n^{O(1)}$.
- If we can solve the problem on tree decomposition of width w in time $w^{O(w)} \cdot n^{O(1)}$, then the running time is $2^{O(\sqrt{k} \log k)} \cdot n^{O(1)}$.

Contraction bidimensionality

Definition

A graph invariant $x(G)$ is **minor-bidimensional** if

- $x(G') \leq x(G)$ for every minor G' of G, and
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Exercise: DOMINATING SET is **not** minor-bidimensional.

Contraction bidimensionality

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Exercise: Dominating Set is **not** minor-bidimensional.

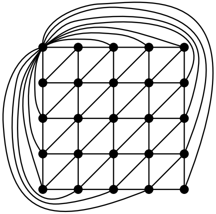
We fix the problem by allowing only contractions but not edge/vertex deletions.

Contraction bidimensionality

Theorem

Every **planar graph** with treewidth at least $9k + 5$ can be contracted to a **triangulated** $k \times k$ grid Γ_k.

5×5 triangulated grid Γ_5:



Contraction bidimensionality

Definition

A graph invariant $x(G)$ is **contraction-bidimensional** if

- $x(G') \leq x(G)$ for every contraction G' of G, and
- $x(\Gamma_k) \geq ck^2$ (for some $c > 0$).

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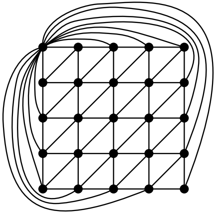

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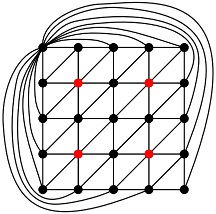

Example: maximum independent set, minimum dominating set are contraction-bidimensional.

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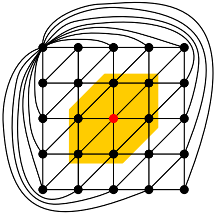


Example: maximum independent set, minimum dominating set are contraction-bidimensional.

Bidimensionality for DOMINATING SET

The size of a minimum dominating set is a **contraction bidimensional** invariant: we need at least $(k-2)^2/7$ vertices to dominate all the internal vertices of the triangulated $k \times k$ grid Γ_k (since a vertex can dominate at most 7 internal vertices).

Theorem

Given a tree decomposition of width w, DOMINATING SET can be solved in time $3^w \cdot w^{O(1)} \cdot n^{O(1)}$.

Solving DOMINATING SET on planar graphs:

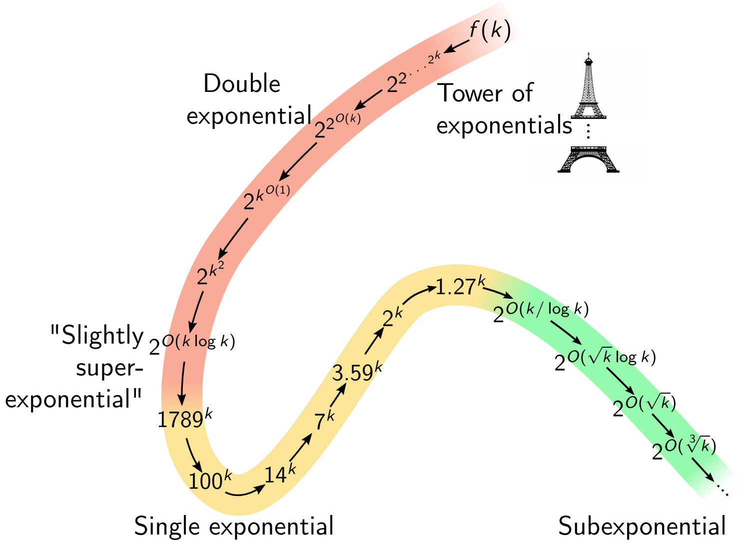
- Set $w := 9(3\sqrt{k}+2)$.
- Use the 4-approximation tree decomposition algorithm.
  - If treewidth is at least w: we answer 'dominating set is $\geq k$'.
  - If we get a tree decomposition of width $4w$, then we can solve the problem in time $3^w \cdot n^{O(1)} = 2^{O(\sqrt{k})} \cdot n^{O(1)}$.

Beyond bidimensionality

The following problems are **not** minor/contraction-bidimensional, but admit $2^{O(\sqrt{k} \cdot \text{polylog} k)} \cdot n^{O(1)}$ time algorithms on planar graphs using other techniques:

- Odd Cycle Transversal
- k-Path in directed planar graphs
- Subset Feedback Vertex Set
- Multiway Cut
- Subset TSP (parameterized by the number of terminals)
- ...

The race for better FPT algorithms

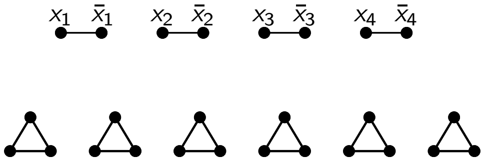


Lower bounds based on ETH

Exponential Time Hypothesis (ETH) + Sparsification Lemma

There is no $2^{o(n+m)}$-time algorithm for n-variable m-clause 3SAT.

The textbook reduction from 3SAT to Vertex Cover:



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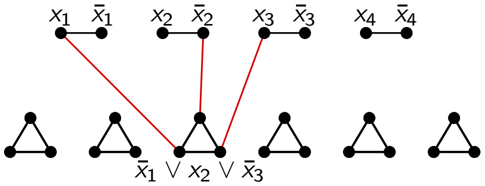
formula is satisfiable $\Leftrightarrow$ there is a vertex cover of size $n+2m$



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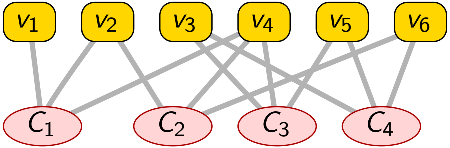
3SAT formula ϕ n variables m clauses	$\Rightarrow$	Graph G $O(n+m)$ vertices $O(n+m)$ edges



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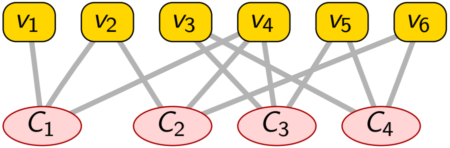

Corollary

Assuming ETH, there is no $2^{o(n)}$ algorithm for Vertex Cover on an n-vertex graph.

Lower bounds based on ETH

Exponential Time Hypothesis (ETH) + Sparsification Lemma

There is no $2^{o(n+m)}$-time algorithm for n-variable m-clause 3SAT.

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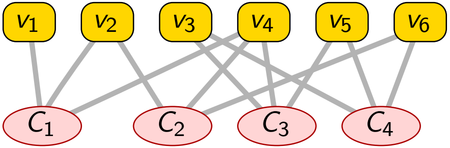


Corollary

Assuming ETH, there is no $2^{o(k)} \cdot n^{O(1)}$ algorithm for Vertex Cover.

Other problems

There are polytime reductions from 3SAT to many problems such that the reduction creates a graph with $O(n+m)$ vertices/edges.

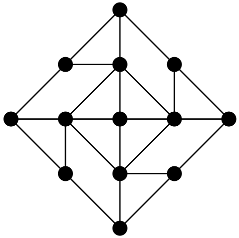
Consequence: Assuming ETH, the following problems cannot be solved in time $2^{o(n)}$ and hence in time $2^{o(k)} \cdot n^{O(1)}$ (but $2^{O(k)} \cdot n^{O(1)}$ time algorithms are known):

- Vertex Cover
- Longest Cycle
- Feedback Vertex Set
- Multiway Cut
- Odd Cycle Transversal
- Steiner Tree
- ...

Lower bounds based on ETH

What about 3-Coloring on planar graphs?

The textbook reduction from 3-Coloring to Planar 3-Coloring uses a "crossover gadget" with 4 external connectors:



- In every 3-coloring of the gadget, opposite external connectors have the same color.
- Every coloring of the external connectors where the opposite vertices have the same color can be extended to the whole gadget.
- If two edges cross, replace them with a crossover gadget.

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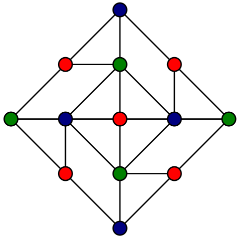

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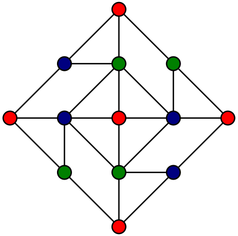
Lower bounds based on ETH

What about 3-Coloring on planar graphs?

The textbook reduction from 3-Coloring to Planar 3-Coloring uses a "crossover gadget" with 4 external connectors:



- In every 3-coloring of the gadget, opposite external connectors have the same color.
- Every coloring of the external connectors where the opposite vertices have the same color can be extended to the whole gadget.
- If two edges cross, replace them with a crossover gadget.

Lower bounds based on ETH

- The reduction from 3-Coloring to Planar 3-Coloring introduces $O(1)$ new edges/vertices for each crossing.
- A graph with m edges can be drawn with $O(m^2)$ crossings.

3SAT formula ϕ n variables m clauses	$\Rightarrow$	Graph G $O(m)$ vertices $O(m)$ edges	$\Rightarrow$	Planar graph G' $O(m^2)$ vertices $O(m^2)$ edges

Corollary

Assuming ETH, there is no $2^{o(\sqrt{n})}$ algorithm for 3-Coloring on an n-vertex planar graph G.

Lower bounds for planar problems

Consequence: Assuming ETH, there is no $2^{o(\sqrt{n})}$ time algorithm on n-vertex **planar graphs** for

- Independent Set
- Dominating Set
- Vertex Cover
- Hamiltonian Path
- Feedback Vertex Set
- ...

Lower bounds for planar problems

Consequence: Assuming ETH, there is no $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithm on **planar graphs** for

- Independent Set
- Dominating Set
- Vertex Cover
- k-Path
- Feedback Vertex Set
- ...

An exceptional lower bound

STEINER TREE: Given a graph G and set T of terminals, find a tree of minimum size that contains T.

We have seen:

- Can be solved in time $3^{|T|} \cdot n^{O(1)}$ using dynamic programming.
- Can be solved in time $2^{|T|} \cdot n^{O(1)}$ using algebraic techniques.

Is there a subexponential FPT algorithm on planar graphs?

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Theorem

Assuming ETH, STEINER TREE on planar graphs with k terminals cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$.

Treewidth — summary

- Notion of treewidth: widely used in graph theory and parameterized algorithms.
- Efficient algorithms parmeterized by treewidth.
- Applications e.g. to planar graphs.

Treewidth

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If u and v are neighbors, then there is a bag containing both of them.
2. For every v, the bags containing v form a connected subtree.

Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.