Representative sets and Matroids

Lecture #09

December 14, 2021

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Slides courtesy: Neeldhara Misra, Saket Saurabh, Pranabendu Misra

Problems we would be interested in...

Vertex Cover

Input: A graph $G = (V, E)$ and a positive integer k.

Parameter: k

Question: Does there exist a subset $V' \subseteq V$ of size at most k such that for every edge $(u, v) \in E$ either $u \in V'$ or $v \in V'$?

Hamiltonian Path

Input: A graph $G = (V, E)$

Question: Does there exist a path P in G that spans all the vertices?

K- Path

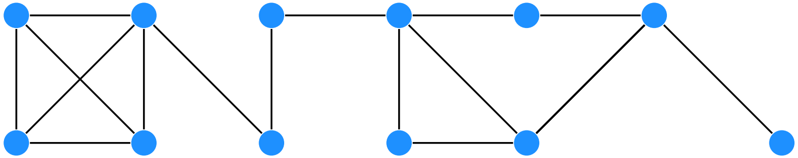
Input: A graph $G = (V, E)$ and a positive integer k.

Parameter: k

Question: Does there exist a path P in G of length at least k?

Dynamic Programming for Hamiltonian Path

∘ Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad n-1 \quad n$

v_1

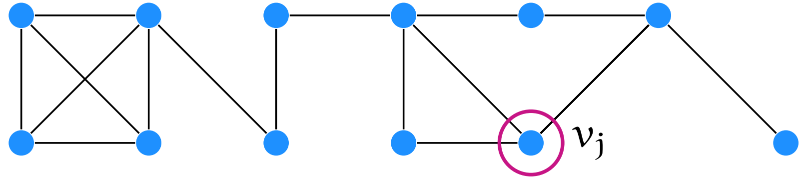
$\vdots$

v_j

$\vdots$

v_n

∘ Ham-Path



1 2 3 $\cdots$ $\boxed{i}$ $\cdots$ $n-1$ n

v_1

$\vdots$

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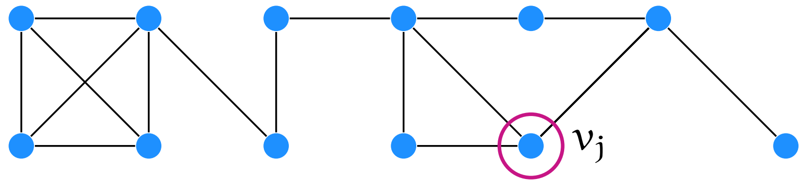
v_n

Ham-Path



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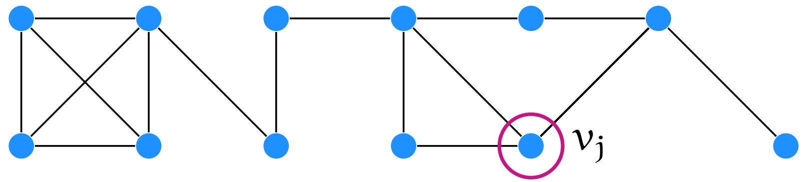
∘ Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

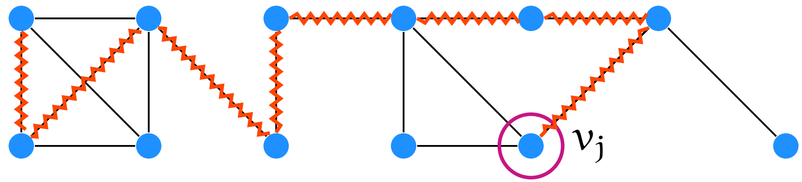
$\vdots$

v_j $\qquad$ V[Paths of length i ending at v_j]

$\vdots$

v_n

∘ Ham-Path

Example: 

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v_n

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∘ Ham-Path

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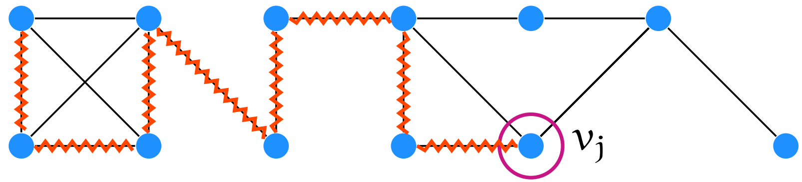

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v_1

$\vdots$

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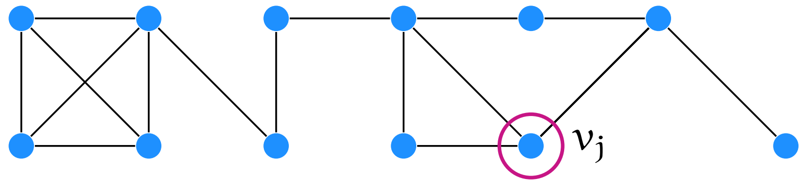
$\vdots$

v_n

Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

$\vdots$

v_j

$\vdots$

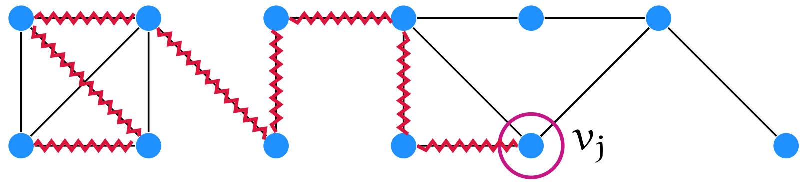
v_n

SETS, NOT SEQUENCES.

$V[\overbrace{\text{Paths of length } i \text{ ending at } v_j}]$

Ham-Path

Example:



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

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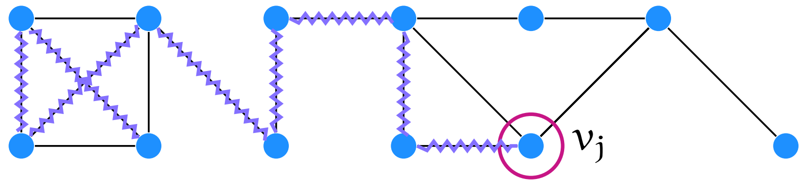
v_n

$$\overbrace{V[\text{Paths of length } i \text{ ending at } v_j]}^{\text{sets, not sequences.}}$$

Ham-Path

Example:



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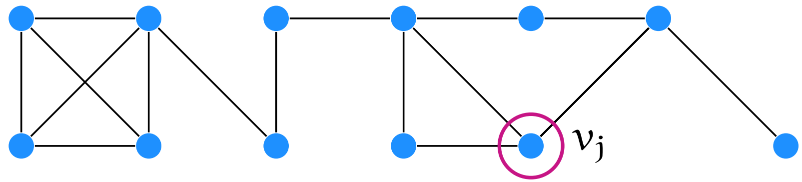

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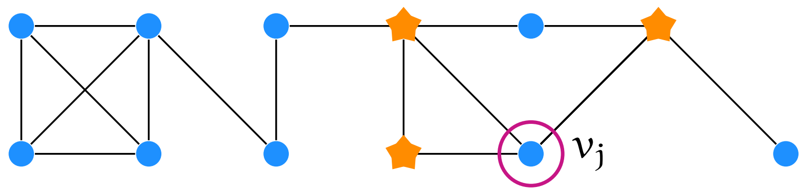
$$\overbrace{V[\text{Paths of length } i \text{ ending at } v_j]}^{\text{SETS, NOT SEQUENCES.}}$$

Two paths that use the same set of vertices but visit them in different orders are equivalent.

∘ Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

$\vdots$

v_j

$\vdots$

v_n

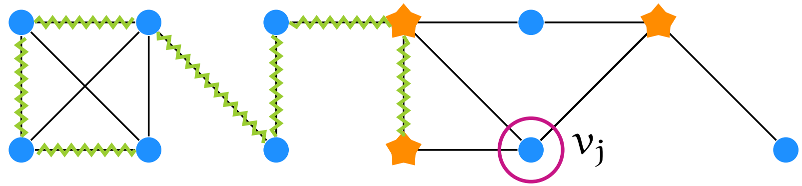
V[Paths of length i ending at v_j]

= V[Paths of length $(i-1)$ ending at u, avoiding v_j.]

$\forall\ u \in N(v_j)$

Ham-Path

Valid:



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v_1

$\vdots$

v_j

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v_n

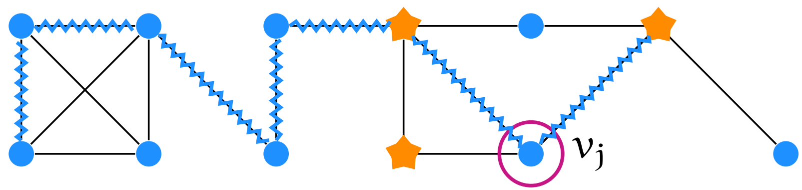
V[Paths of length i ending at v_j]

$=$ V[Paths of length $(i-1)$ ending at u, avoiding v_j.]

$\forall u \in N(v_j)$

Ham-Path

Invalid:



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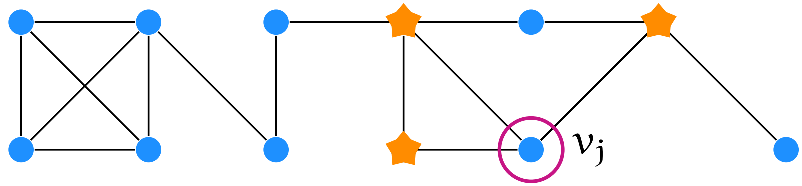
$\forall u \in N(v_j)$

v_n

∘ Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

$\vdots$

Potentially storing $\binom{n}{i}$ sets.

v_j

V[Paths of length i ending at v_j]

$\vdots$

= V[Paths of length $(i-1)$ $\boxed{\text{ending at } u,}$ avoiding v_j .]

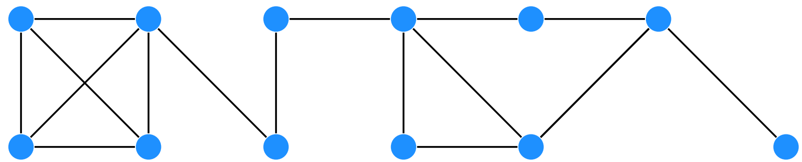
$\forall u \in N(v_j)$

v_n

Let us now turn to k-Path.

To find paths of length at least k,
we may simply use the DP table for Hamiltonian Path
restricted to the first k columns.

K-Path



$$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$$

v_1

$\vdots$

v_j

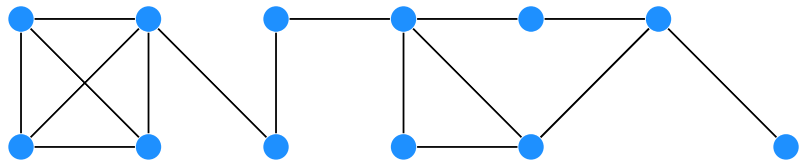
Worst case running time: $\mathcal{O}^\star\left(\binom{n}{k}\right)$ if $k \leq n/2$

$\mathcal{O}^*\left(\binom{n}{n-k}\right)$ if $k > \frac{n}{2}$

$\vdots$

v_n

K-Path



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

$\vdots$

v_j

Worst case running time: $\mathcal{O}^\star(n^k)$

$\vdots$

v_n

Do we really need to store all these sets?

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In the i^{th} column, we are storing paths of length i.

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There may be several paths of length i that "latch on" to the last $(k - i)$ vertices of P.

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In the i^{th} column, we are storing paths of length i.

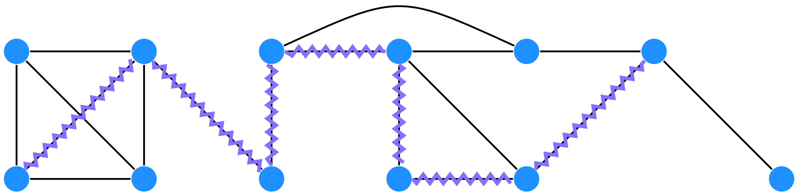
Let P be a path of length k.

There may be several paths of length i that "latch on" to the last $(k-i)$ vertices of P.

We need to store just one of them.

Example.

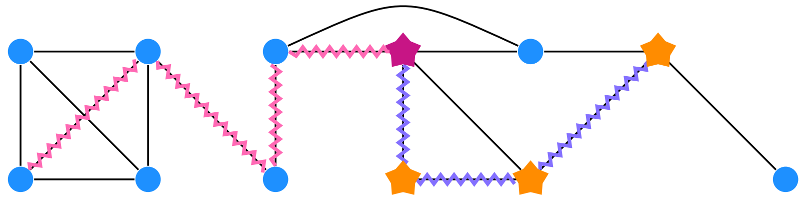
Suppose we have a path P on seven edges.

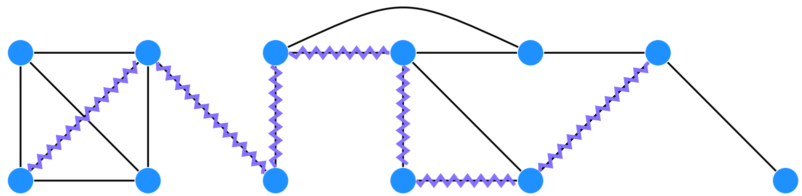


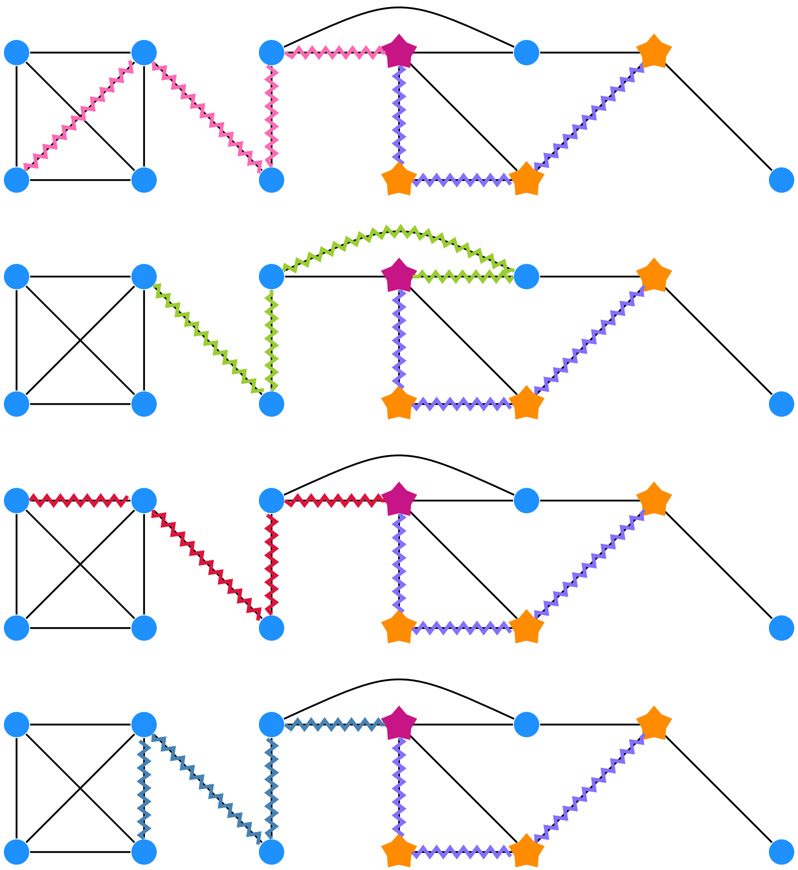
Example.

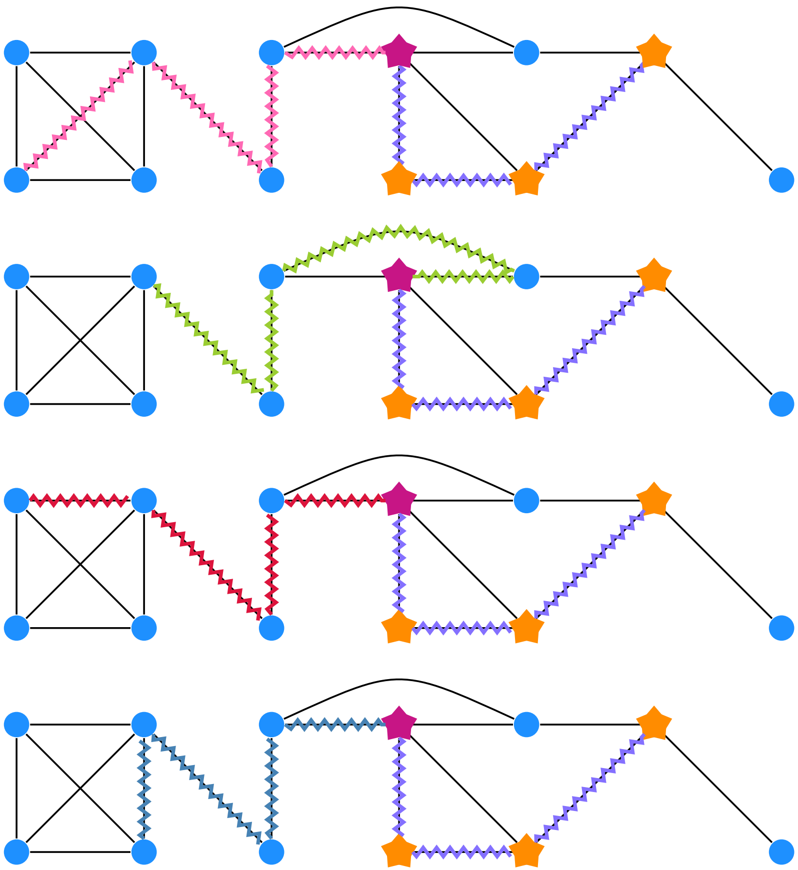
Suppose we have a path P on seven edges.

Consider it broken up into the first four and the last three edges.



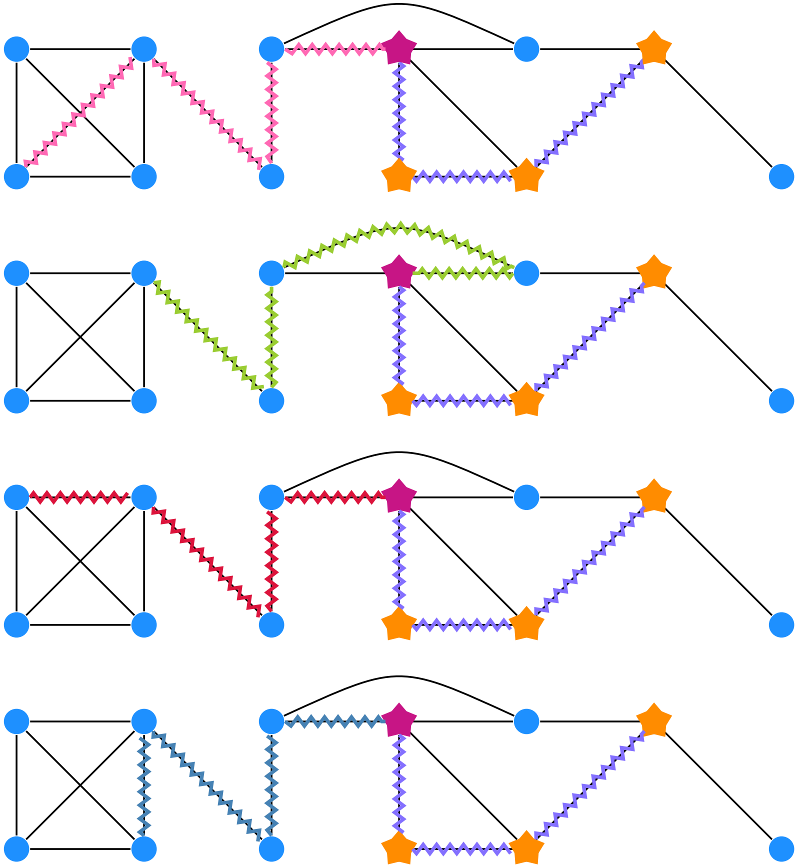






A Fixed Future ($v_{i+1} - \cdots - v_k$).

The Possibilities for Partial Solutions Compatible with $v_{i+1} - \cdots - v_k$.



A Fixed Future ($v_{i+1} - \cdots - v_k$).

For any possible ending of length $(k - i)$, we want to be sure that we store at least one among the possibly many “prefixes”.

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This could also be $\binom{n}{k-i}$.

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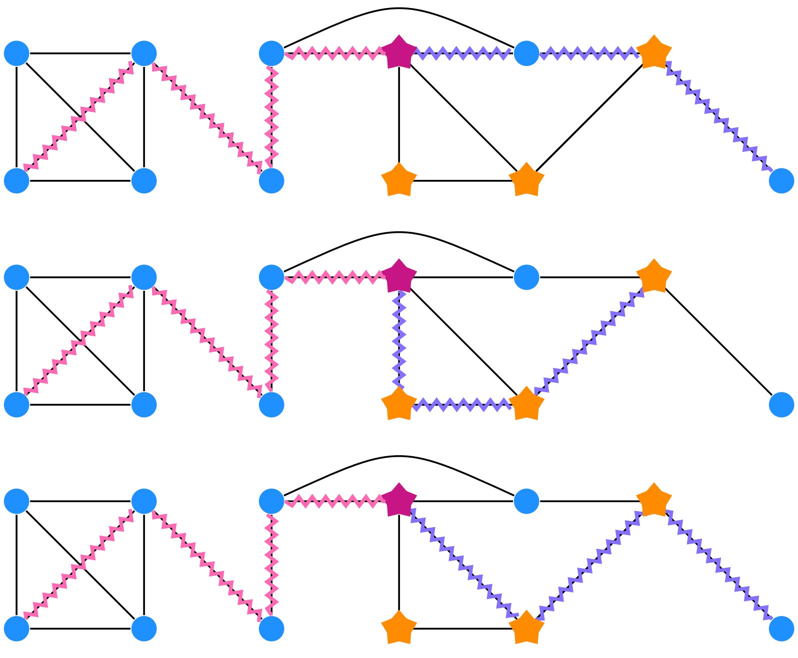
This could also be $\binom{n}{k-i}$.

The hope for "saving" comes from the fact that a single path of length i is potentially capable of being a prefix to several distinct endings.

For example...



REPRESENTATIVE SETS

Partial solutions: paths of length j ending at v_i

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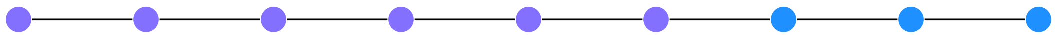
A "small" representative family.

If:



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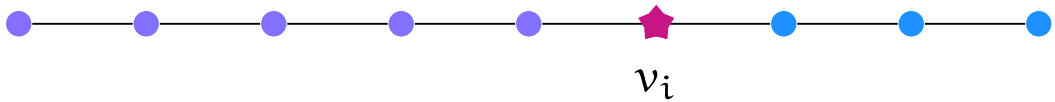

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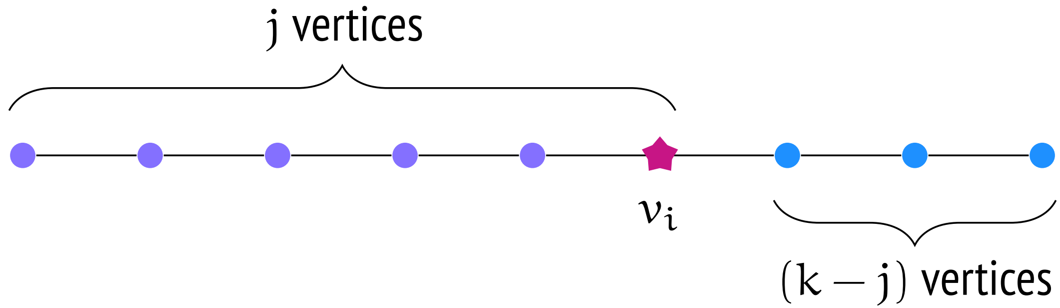

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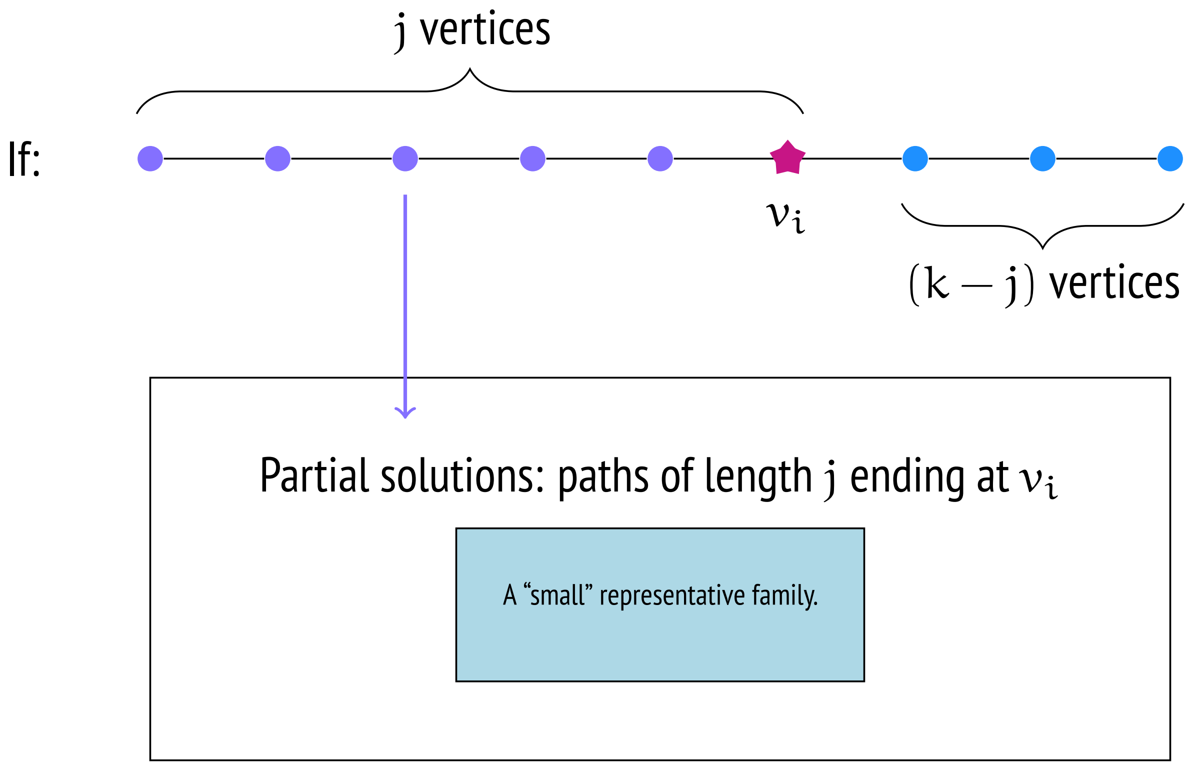
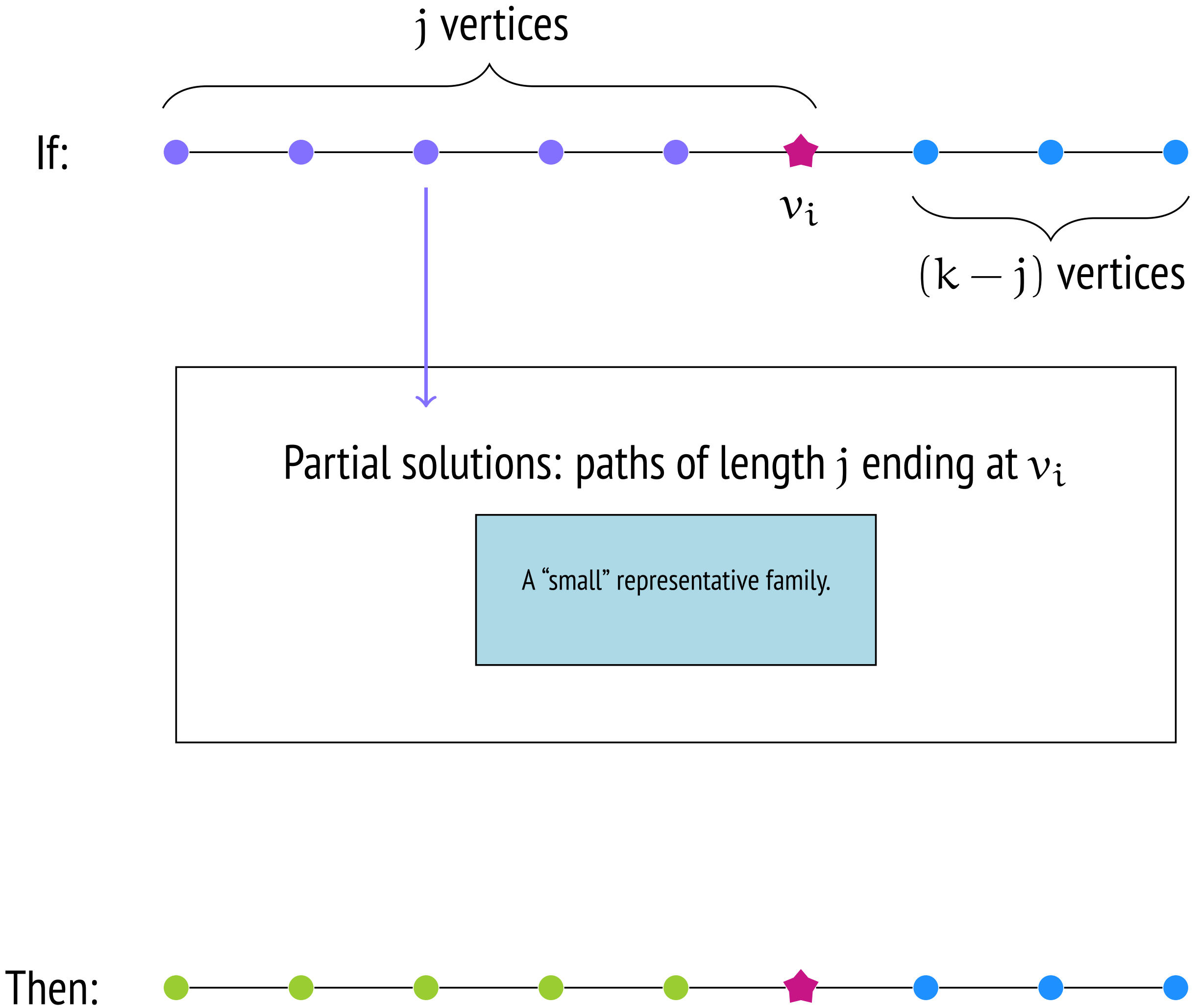

If:
j vertices
v_i
$(k-j)$ vertices
Partial solutions: paths of length j ending at v_i
A “small” representative family.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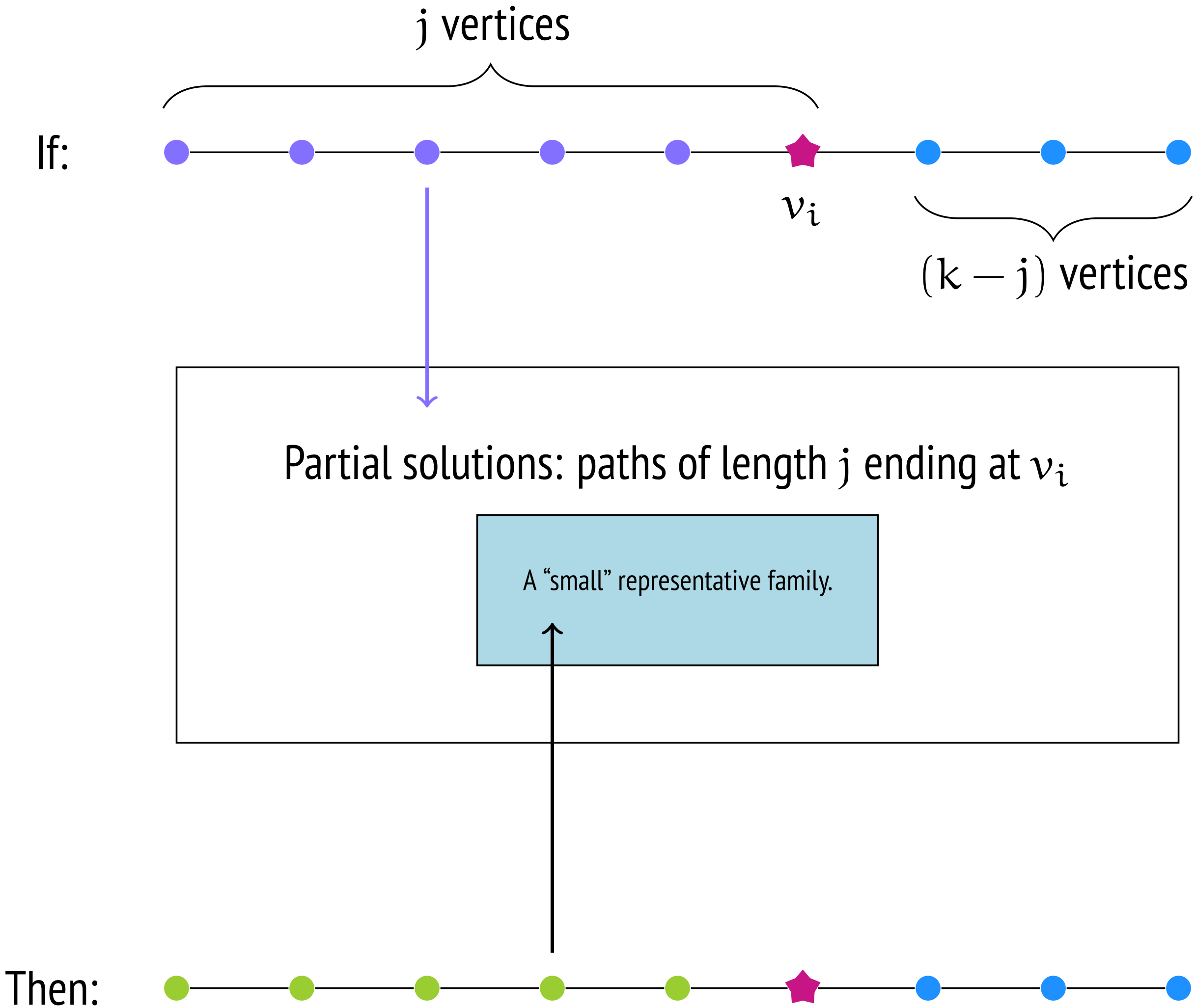
Then:



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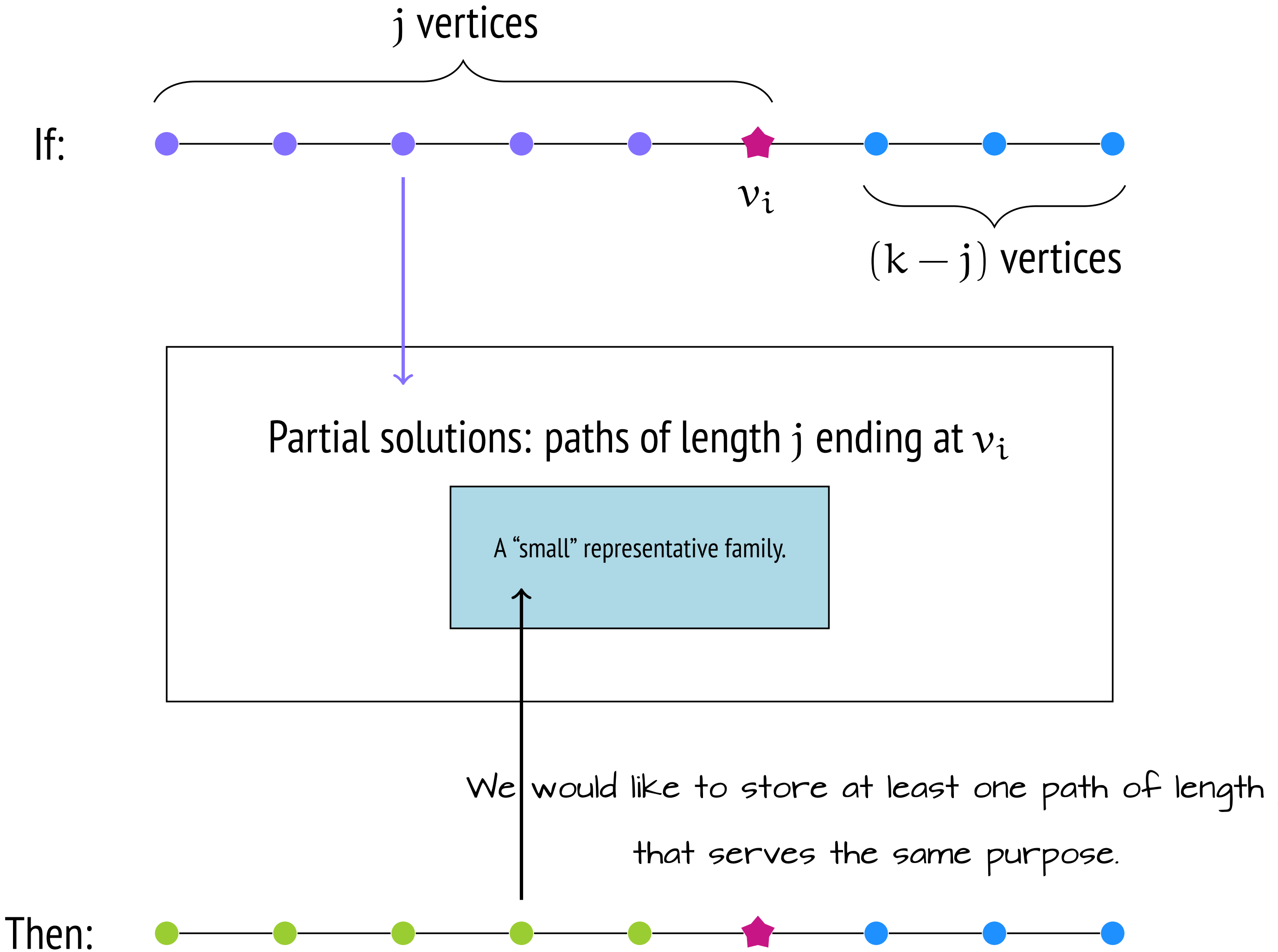

j vertices
If:
v_i
$(k - j)$ vertices
Partial solutions: paths of length j ending at v_i
A "small" representative family.
We would like to store at least one path of length j
that serves the same purpose.
Then:

Given: A (BIG) family $\mathcal{F}$ of p-sized subsets of $[n]$.

$$S_1, S_2, \ldots, S_t$$

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Want: A (small) subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

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Want: A (small) subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k-p)$,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$.

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The “second half” of a solution – can be any subset.

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This is a valid patch into X.

Given: A (BIG) family $\mathcal{F}$ of p-sized subsets of $[n]$.

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This is a guaranteed replacement for S.

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Given: A $\leqslant \binom{n}{p}$ family $\mathcal{F}$ of p-sized subsets of $[n]$.

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Given: $A \leqslant \binom{n}{p}$ family $\mathcal{F}$ of p-sized subsets of $[n]$.

$$S_1, S_2, \ldots, S_t$$

Known: $\exists \binom{k}{p}$ subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k-p)$,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$.

Bolobás, 1965.

Given: A a matroid $(E, \mathcal{I})$, and a family of p-sized subsets from $\mathcal{I}$:

$$S_1, S_2, \ldots, S_t$$

There is a subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ of size at most $\binom{p+q}{p}$ such that:

For any $X \subseteq [n]$ of size at most q,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$ and $X \cup S \in \mathcal{I}$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$ and $X \cup \widehat{S} \in \mathcal{I}$.

Lovász, 1977

representable

Given: A a matroid $(E, \mathcal{I})$, and a family of p-sized subsets from $\mathcal{I}$:

$$S_1, S_2, \ldots, S_t$$

There is an efficiently computable subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ of size at most $\binom{p+q}{p}$ such that:

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Márx (2009) and Fomin, Lokshtanov, Saurabh (2013)

Matroids

Definition

A pair $M = (E, \mathcal{I})$, where E is a ground set and $\mathcal{I}$ is a family of subsets (called independent sets) of E, is a *matroid* if it satisfies the following conditions:

(I1) $\phi \in \mathcal{I}$
(I2) If $A' \subseteq A$ and $A \in \mathcal{I}$ then $A' \in \mathcal{I}$.
(I3) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $\exists\, e \in (B \setminus A)$ such that $A \cup \{e\} \in \mathcal{I}$.

The axiom (I2) is also called the *hereditary property* and a pair $M = (E, \mathcal{I})$ satisfying (I1) and (I2) is called *hereditary family* or *set-family*.

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(I3) is called the exchange property.

Rank and Basis

Definition

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An inclusion wise maximal set of $\mathcal{I}$ is called a *basis* of the matroid. Using axiom (I3) it is easy to show that all the bases of a matroid have the same size. This size is called the *rank* of the matroid M, and is denoted by $\mathsf{rank}(M)$.

Examples Of Matroids

Uniform Matroid

A pair $M = (E, \mathcal{I})$ over an n-element ground set E, is called a *uniform matroid* if the family of independent sets $\mathcal{I}$ is given by

$$\mathcal{I} = \Big\{ A \subseteq E \mid |A| \leq k \Big\},$$

where k is some constant. This matroid is also denoted as $U_{n,k}$.

Eg: $E = \{1, 2, 3, 4, 5\}$ and $k = 2$ then

$$\begin{aligned} \mathcal{I} \;=\; & \Big\{ \{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \\ & \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\} \Big\} \end{aligned}$$

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Graphic Matroid

Given a graph G, a *graphic matroid* is defined as $M = (E, \mathcal{I})$ where and

- $E = E(G)$ – edges of G are elements of the matroid
- $$\mathcal{I} = \left\{ F \subseteq E(G) : \ F \text{ is a forest in the graph } G \right\}$$

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Exchange property?

Given: $A, B \in \mathcal{I}$, $|A| < |B|$

To show: $\exists\ e \in B$ such that $A \cup \{e\} \in \mathcal{I}$

Proof:

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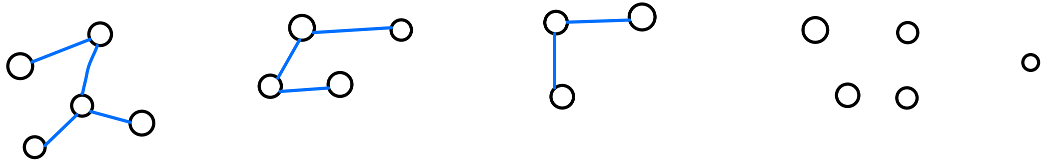

Matroid Representation

Remark

- Need compact representation to for the family of independent sets.
- Also to be able to test easily whether a set belongs to the family of independent sets.

Linear Matroid

Let A be a matrix over an arbitrary field $\mathbb{F}$ and let E be the set of columns of A. Given A we define the matroid $M = (E, \mathcal{I})$ as follows.
A set $X \subseteq E$ is independent (that is $X \in \mathcal{I}$) if the corresponding columns are *linearly independent* over $\mathbb{F}$.

$$A = \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \quad * \text{ are elements of } F$$

The matroids that can be defined by such a construction are called *linear matroids*.

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Linear Matroids and Representable Matroids

If a matroid can be defined by a matrix A over a field $\mathbb{F}$, then we say that the matroid is *representable* over $\mathbb{F}$.

Linear Matroids and Representable Matroids

A matroid $M = (E, \mathcal{I})$ is representable over a field $\mathbb{F}$ if there exist vectors in $\mathbb{F}^\ell$ that correspond to the elements such that the linearly independent sets of vectors precisely correspond to independent sets of the matroid.

Let $E = \{e_1, \ldots, e_m\}$ and ℓ be a positive integer.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ \ell \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}_{\ell \times m} \end{array}$$

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$$
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\end{array}
$$

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Uniform Matroid $\mathcal{U}_{n,k}$

Every uniform matroid is linear and can be represented over a finite field by a $k \times n$ matrix A_M where the $A_M[i,j] = j^{i-1}$.

$$\begin{array}{cc} & \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ k \end{array} & \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2^{k-1} & 3^{k-1} & \cdots & n^{k-1} \end{bmatrix}_{k\times n} \end{array}$$

Observe that for A_M to be representable over a finite field $\mathbb{F}$, we need that the determinant of any $k \times k$ submatrix of A_M must not vanish over $\mathbb{F}$.

The determinant of any $k \times k$ submatrix of A_M is upper bounded by $k! \times n^{k-1}$ (this follows from the Laplace expansion of determinants). Thus, choosing a field $\mathbb{F}$ of size larger than $k! \times n^{k-1}$ suffices.

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$$
\begin{array}{c c}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_n \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ k \end{array} &
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 2 & 3 & \cdots & n \\
1 & 2^2 & 3^2 & \cdots & n^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2^{k-1} & 3^{k-1} & \cdots & n^{k-1}
\end{bmatrix}_{k \times n}
\end{array}
$$

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Uniform Matroid: Size of the representation

$$
\begin{array}{c c}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_n \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ k \end{array} & \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2^{k-1} & 3^{k-1} & \cdots & n^{k-1} \end{bmatrix}_{k \times n}
\end{array}
$$

So the size of the representation: $O((k \log n) \times nk)$ bits.

Graphic Matroid

The graphic matroid is representable over any field of size at least 2.

Consider the matrix A_M with a row for each vertex $i \in V(G)$ and a column for each edge $e = ij \in E(G)$. In the column corresponding to $e = ij$, all entries are 0, except for a 1 in i or j (arbitrarily) and a -1 in the other.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} &
\begin{bmatrix}
1 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
-1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & -1 & \cdots & -1
\end{bmatrix}_{n \times |E(G)|}
\end{array}
$$

This is a representation over reals.

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$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1^{-1} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1^{-1} & 1^{-1} & \cdots & 1^{-1} \end{array} \right]_{n \times |E(G)|} \end{array}$$

To obtain a representation over a field $\mathbb{F}$, one simply needs to take the representation given above over reals and simply replace all -1 by the additive inverse of 1

Given: A a representable matroid $(E, \mathcal{I})$, and a family $\mathcal{F}$ of p-sized subsets from $\mathcal{I}$:

$$S_1, S_2, \ldots, S_t$$

There is an efficiently computable subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ of size at most $\binom{p+q}{p}$ such that:

For any $X \subseteq [n]$ of size at most q,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$ and $X \cup S \in \mathcal{I}$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$ and $X \cup \widehat{S} \in \mathcal{I}$.

Márx (2009) and Fomin, Lokshtanov, Saurabh (2013)

Summary.

We have at hand a p-uniform collection of independent sets, $\mathcal{F}$ and a number q. Let X be any set of size at most q. For any set $S \in \mathcal{F}$, if:

a X is disjoint from S, and

b X and S together form an independent set,

then a q-representative family $\widehat{\mathcal{F}}$ contains a set $\widehat{S}$ that is:

a disjoint from X, and

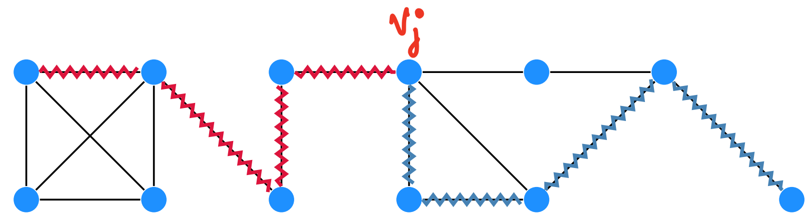
b forms an independent set together with X.

Such a subfamily is called a q-representative family for the given family.

REPRESENTATIVE SETS

and

k-path



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

$\vdots$

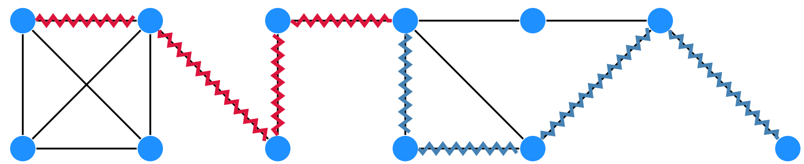
v_j

$\vdots$

v_n

[RECALL]

Worst case running time: $\mathcal{O}^\star\left(\binom{n}{k}\right)$



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

[RECALL]

$\vdots$

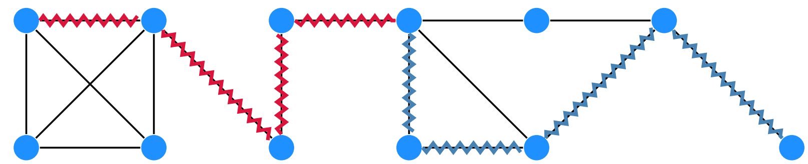
$\binom{n}{i}$

v_j

$\vdots$

v_n

Underlying matroid is the uniform matroid $U_{n,k}$



1 2 3 $\cdots$ i $\cdots$ $k-1$ k

v_1

$\vdots$

v_j

$\vdots$

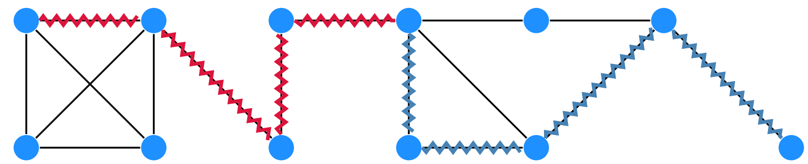
v_n

[RECALL]

$\binom{n}{i}$

Representative Set Computation

Underlying matroid is the uniform matroid $U_{n,k}$



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

$\vdots$

v_j

$\vdots$

v_n

Not so fast!

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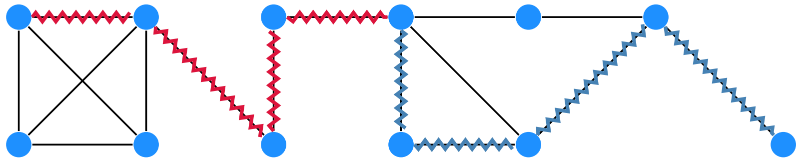
$\binom{n}{i}$

Representative Set Computation

$\binom{k}{i}$



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

Not so fast!

$\vdots$

$\binom{n}{i}$ is too big!

v_j

Representative Set Computation

$\vdots$

v_n

$\binom{k}{i}$

We are going to compute representative families at every intermediate stage of the computation.

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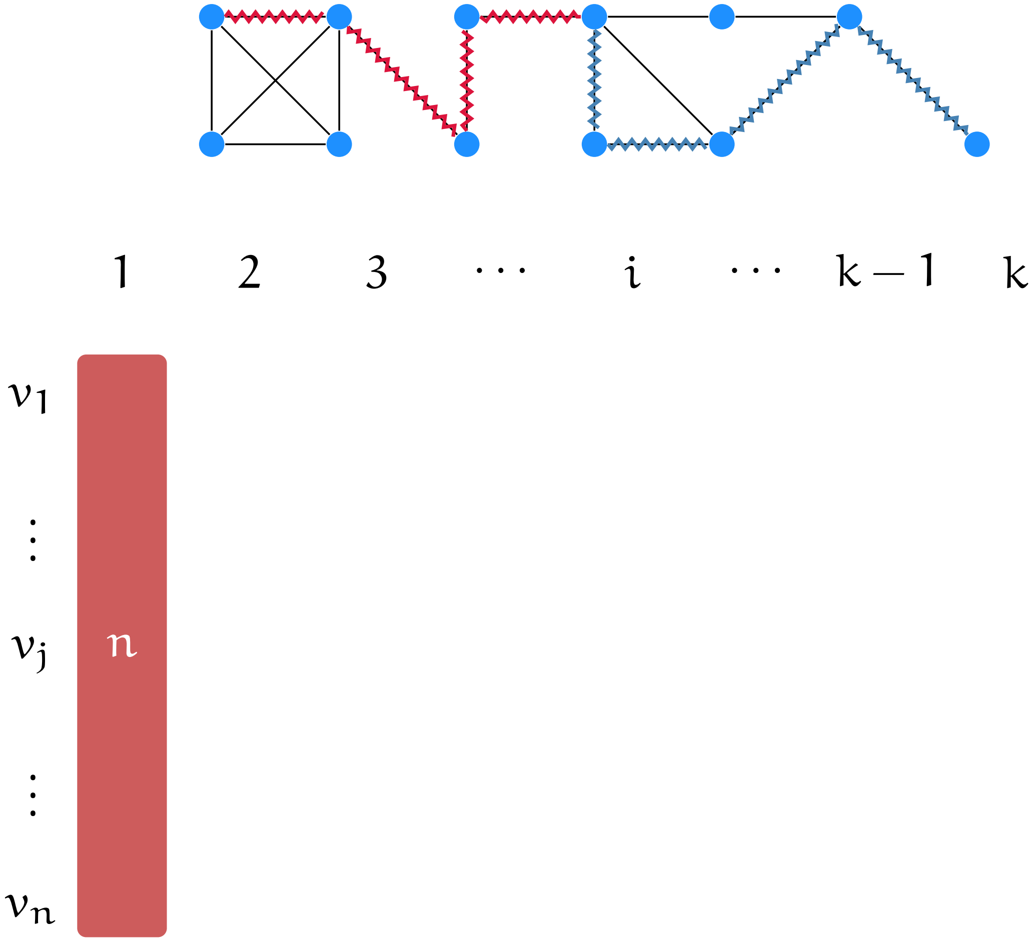
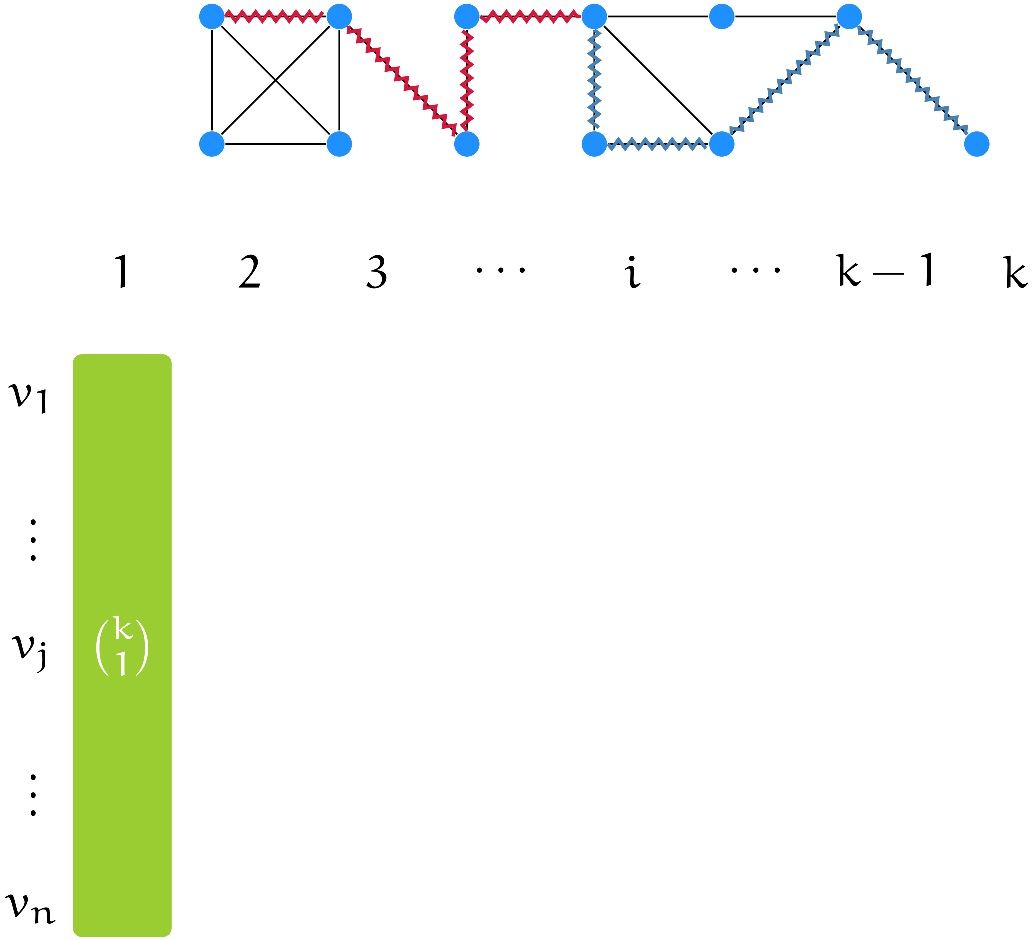
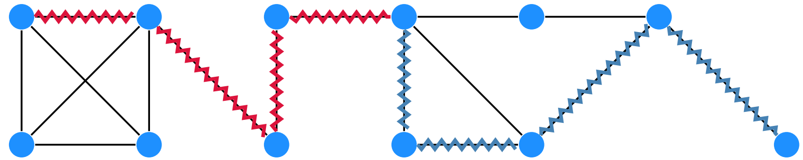
For instance, in the i^{th} column, we are storing i-uniform families.
Before moving on to column $(i+1)$, we compute $(k-i)$-representative families.

This keeps the sizes small as we go along.


1
2
3
· · ·
i
· · ·
k − 1
k
v1
⋮
vj
n
⋮
vn



1 2 3 ⋯ i ⋯ k − 1 k
v_1
⋮
v_j
$\binom{k}{1}$
⋮
v_n



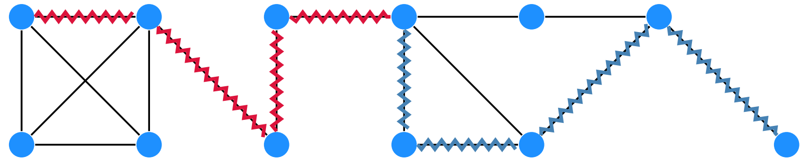
$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

	1	2
v_1		
$\vdots$		
v_j	$\binom{k}{1}$	$\binom{k}{1}n$
$\vdots$		
v_n		

$$V[\text{Paths of length } i \text{ ending at } v_j]$$

$$= V[\text{Paths of length } (i-1) \ \boxed{\text{ending at } u,} \ \text{avoiding } v_j.]$$

$$\forall u \in N(v_j)$$



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

	1	2
v_1		
$\vdots$		
v_j	$\binom{k}{1}$	$\binom{k}{2}$
$\vdots$		
v_n		

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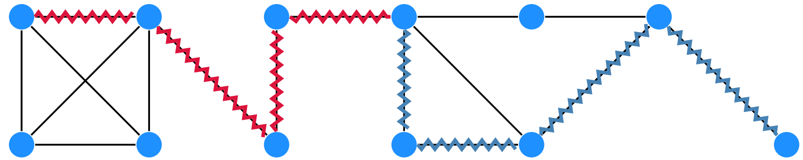

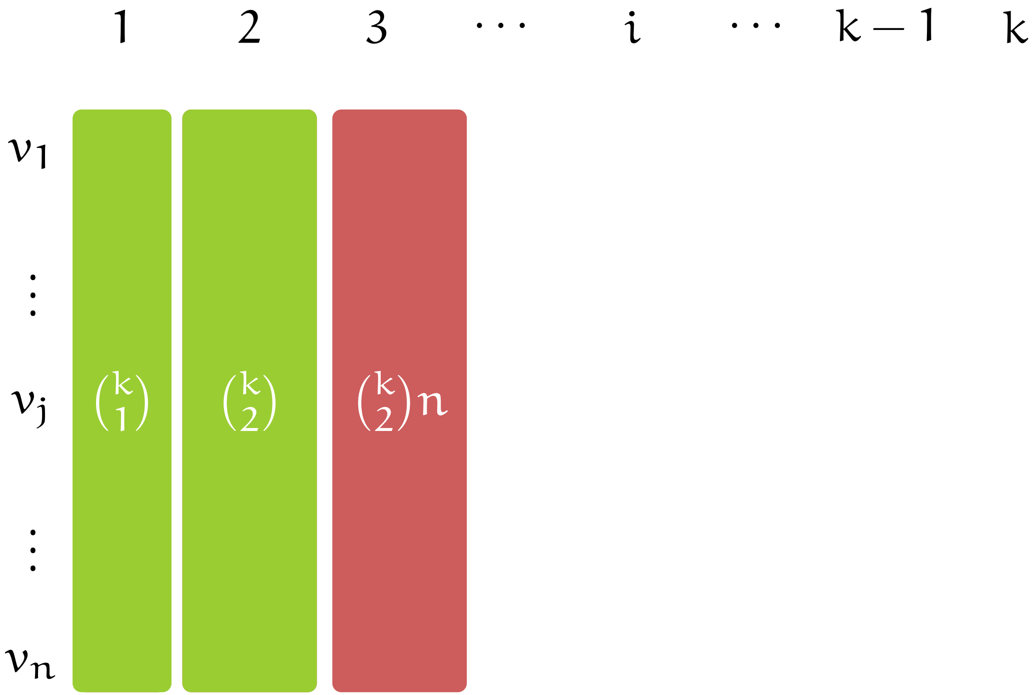

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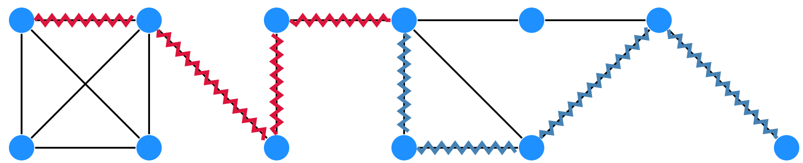

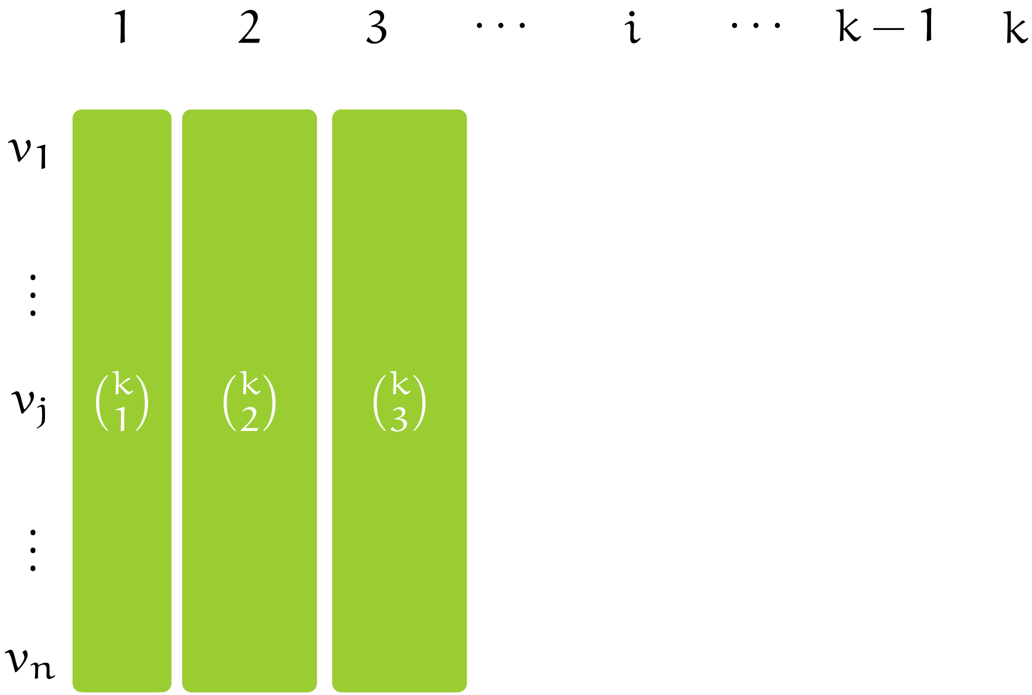

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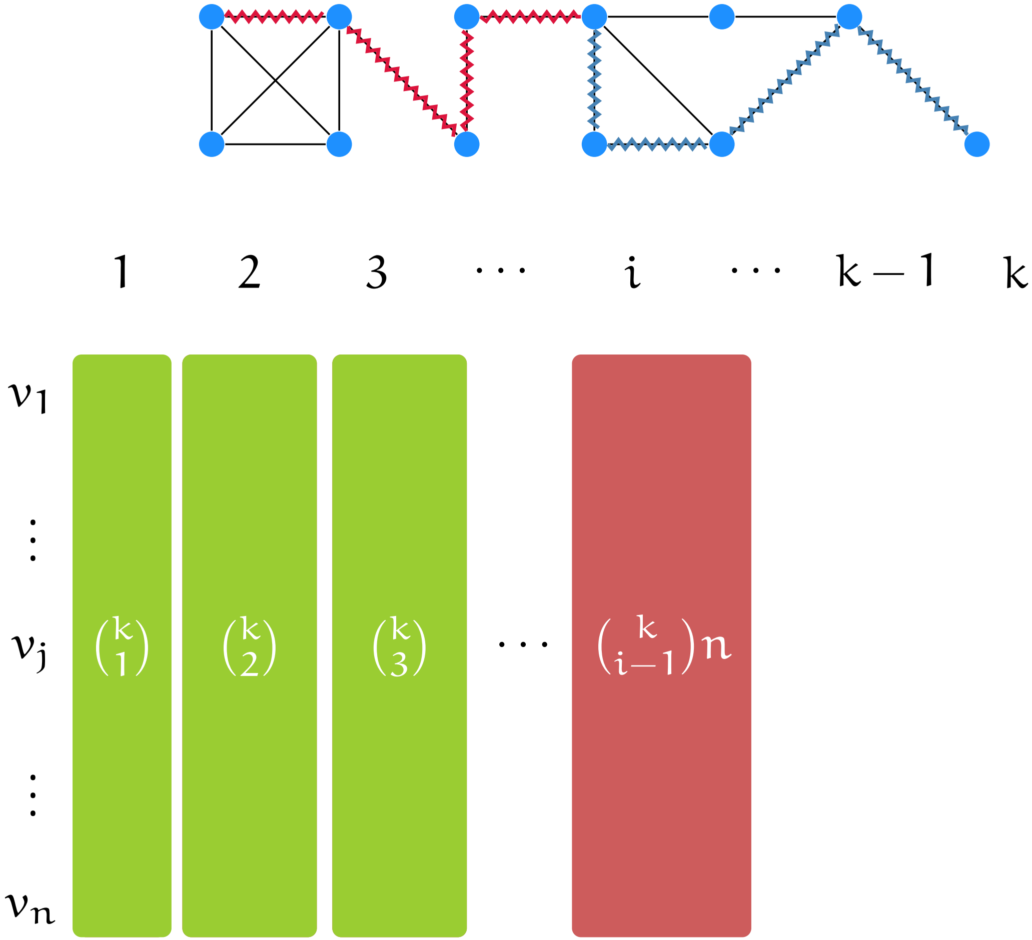
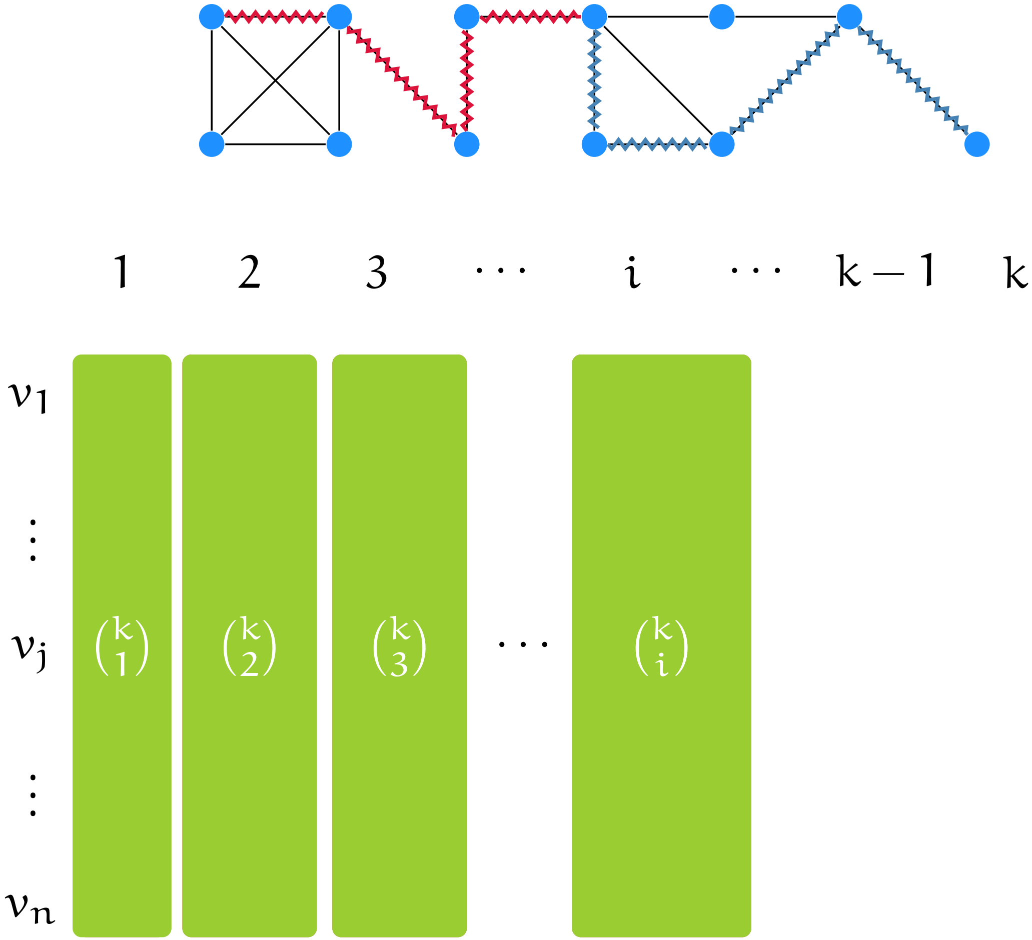
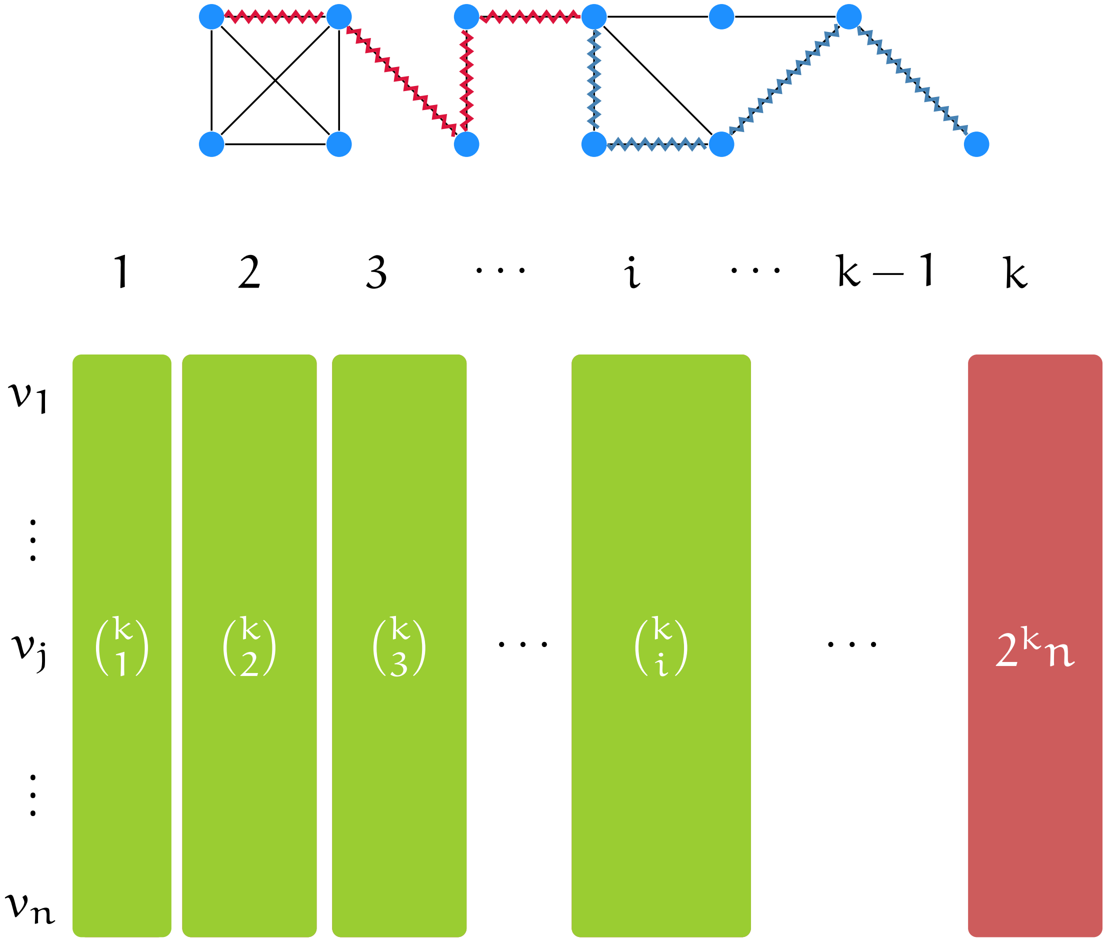


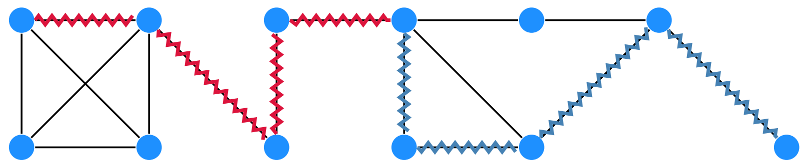
1 2 3 $\cdots$ i $\cdots$ $k-1$ k
v_1
$\vdots$
v_j
$\vdots$
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{3}$
$\cdots$
$\binom{k}{i-1}n$

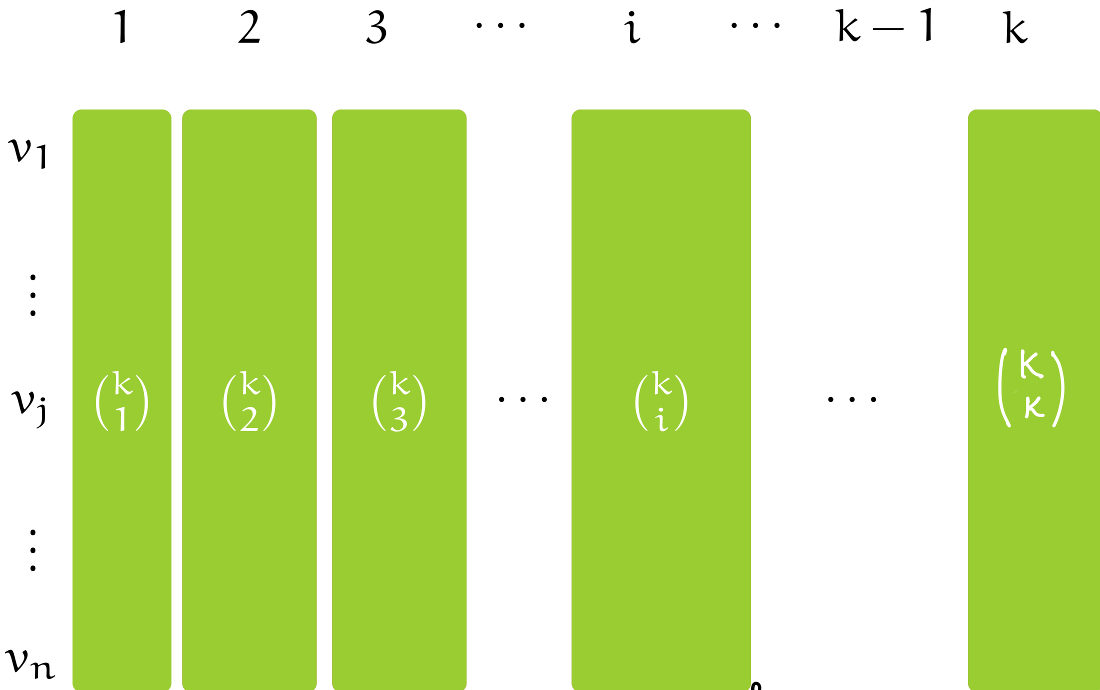


1
2
3
$\cdots$
i
$\cdots$
$k-1$
k
v_1
$\vdots$
v_j
$\vdots$
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{3}$
$\cdots$
$\binom{k}{i}$



1
2
3
⋯
i
⋯
k − 1
k
v_1
⋮
v_j
⋮
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{3}$
⋯
$\binom{k}{i}$
⋯
$2^k n$





$\Rightarrow$ FPT algorithm for k-path

<u>Remark</u>: Using more efficient construction of representative families for uniform matroids, one can design an $O^*(2.619^k)$ algorithm for k-path.

Let $\mathcal{P}_i^j$ be the set of all paths of length i ending at v_j.

It can be shown that the families thus computed at the i^{th} column, j^{th} row are indeed $(k-i)$-representative families for P_i^j.

The correctness is implicit in the notion of a representative family.

REPRESENTATIVE SETS

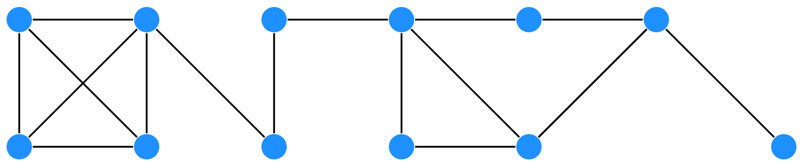
and

Kernelization

Remark: The only known polynomial kernel for the Odd Cycle Transversal problem is via matroids and representative sets.

Vertex Cover

Can you delete k vertices to kill all edges?



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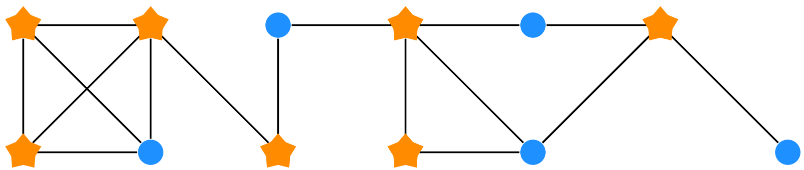

Let $(G = (V, E), k)$ be an instance of Vertex Cover.

Note that E can be thought of as a 2-uniform family over the ground set V.

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Goal: Kernelization.

poly in k

In this context, we are asking if there is a small subset X of the edges such that

$G[X]$ is a YES-instance $\leftrightarrow$ G is a YES-instance.

Note: If G is a YES-instance, then $G[X]$ is a YES-instance for any subset $X \subseteq E$.

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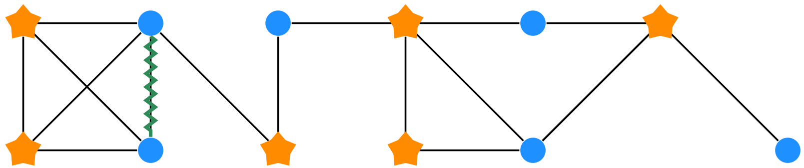
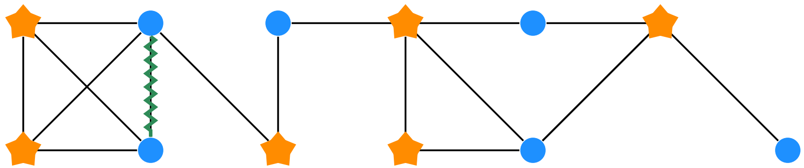
What is a NO-instance?



If G is a NO-instance:

For any subset S of size at most k,
there is an edge that is disjoint from S.



If G is a NO-instance:

For any subset S of size at most k,
there is an edge that is disjoint from S. in $E(G)$

Ring a bell?

Recall.

We have at hand a p-uniform collection of independent sets, $\mathcal{F}$ and a number q. Let X be any set of size at most q. For any set $S \in \mathcal{F}$, if:

a. X is disjoint from S, and
b. X and S together form an independent set,

then a q-representative family contains a set $\widehat{S}$ that is:

a. disjoint from X, and
b. forms an independent set together with X.

Such a subfamily is called a q-representative family for the given family.

Claim: A k-representative family for E is in fact an $O(k^2)$ kernel for vertex cover.

$$\boxed{E(G) = \{e_1, e_2, \ldots, e_m\}}$$

Is there a Vertex Cover of size at most k?

$E(G) = \{e_1, e_2, \ldots, e_m\}$

k-Representative Family

$X = \{f_1, f_2, \ldots, f_r\} \subseteq E(G)$

$\mathcal{O}(k^2)$

Is there a Vertex Cover of size at most k?

$$E(G) = \{e_1, e_2, \ldots, e_m\}$$

$$\equiv$$

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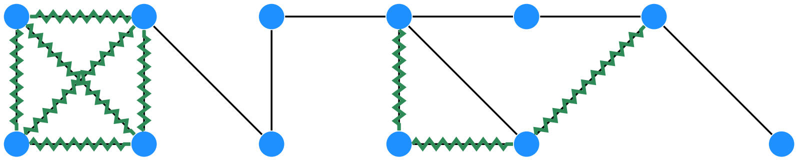
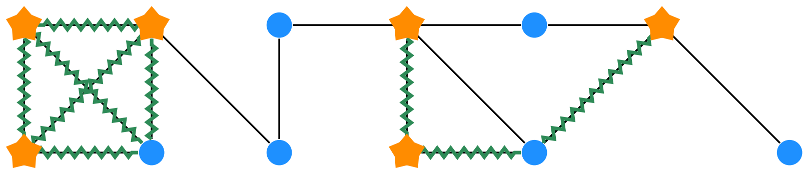
$$\mathcal{O}(k^2)$$

Is there a Vertex Cover of size at most k?

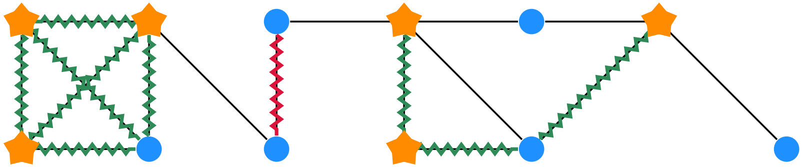
Let us show that if $G[X]$ is a YES-instance, then so is G.

This time, by contradiction.

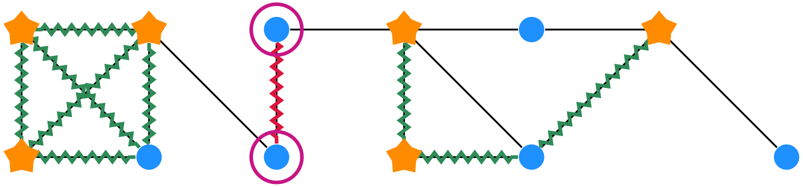




Try the solution for $G[X]$ on G.

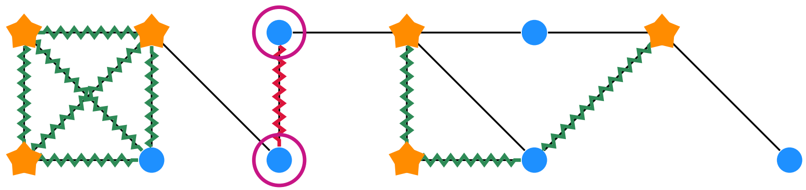


Suppose there is an uncovered edge.



Since X is a k-representative family, for ANY $S \subseteq V$, where $|S| \leqslant k$:

if there is a set e in E such that $e \cap S = \emptyset$,



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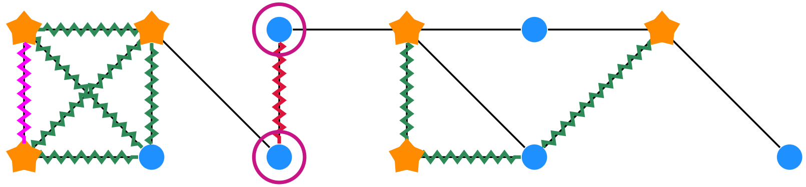
if there is a set e in E such that $e \cap S = \emptyset$,
then there is a set $\widehat{e}$ in X such that $\widehat{e} \cap S = \emptyset$.

Note that the green edges denote $G[X]$.



Since X is a k-representative family, for ANY $\mathrm{S} \subseteq \mathrm{V}$, where $|\mathrm{S}| \leqslant \mathrm{k}$:

if there is a set e in E such that $e \cap S = \varnothing$,
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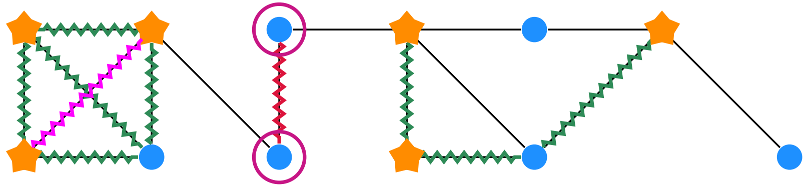
Note that the green edges denote $\mathrm{G}[\mathrm{X}]$.



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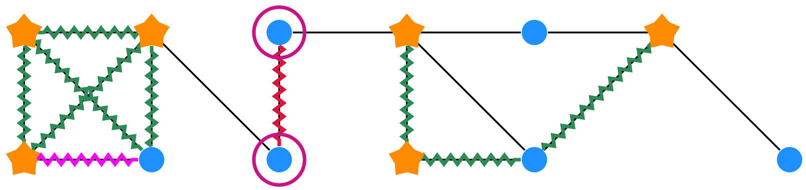

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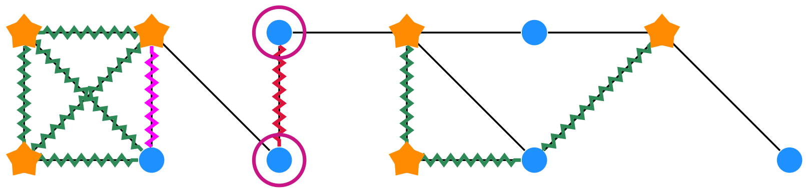

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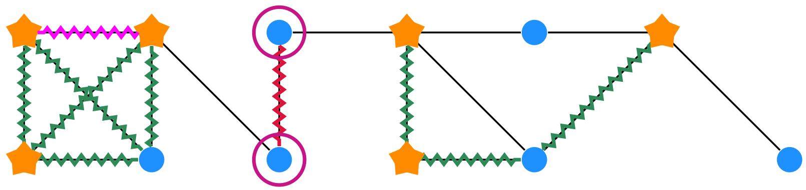

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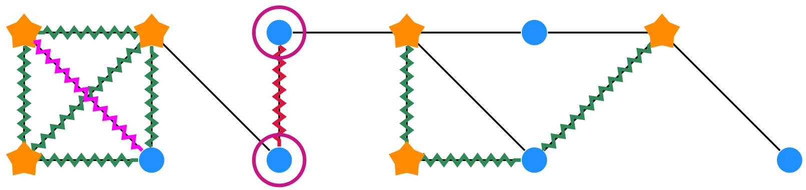

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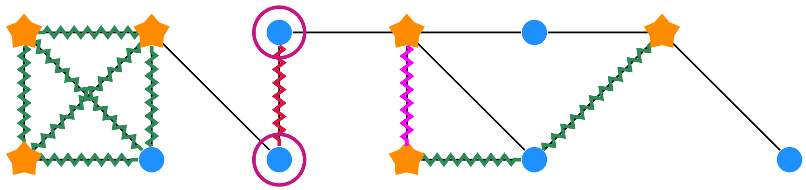

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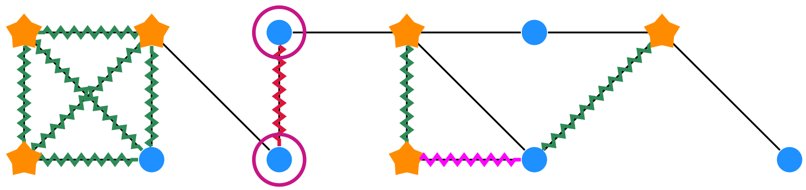

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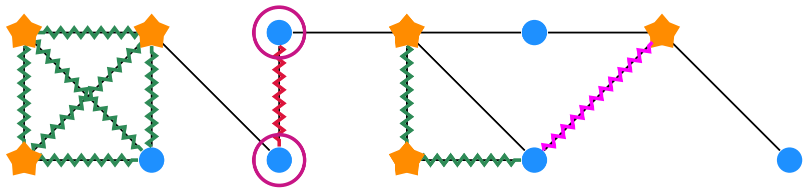

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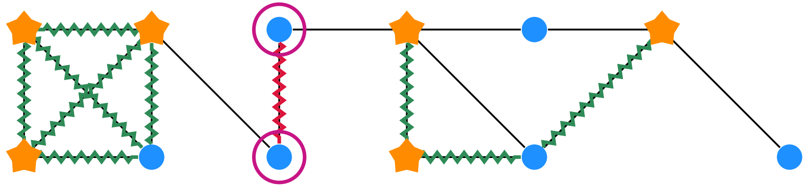

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Contradiction!

A k-representative family for $E(G)$ is in fact an $O(k^2)$ instance kernel for Vertex Cover!