Themistoklis Gouleakis Winter 2021

Property Testing, Exercise Sheet 2

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter20/property-testing

Total Points: 40 Due: Wednesday, **December 2**, 2020

*You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, **using your own words**. Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of all points on exercise sheets to be admitted to the exam. Please send your solutions directly to Philip (`wellnitz@mpi-inf.mpg.de`).*

Exercise 1 — **10** points

Recall the analysis of the linearity tester from the lecture, let $f : G \to H$ denote a function, and let ρ denote the probability that f is rejected by the tester. Further, for a $x \in G$, set $\varphi_y(x) := f(x+y) - f(y)$. Show that the function $\varphi(x) := \operatorname{argmax}_{v \in H}\{|\{y \in G : \varphi_y(x) = v\}|\}$ is a homomorphism.

Hint: Show that the we have $p_{x,y} := \mathsf{Pr}_{r\in G}[\,\varphi(x) + \varphi(y) \neq \varphi(x+y)\,] < 1$; noting that this implies that $p_{x,y} = 0$, as it can only take values in $\{0, 1\}$. You can start by showing that

$$p_{x,y} \le \mathsf{Pr}_{r\in G}[\,(\varphi(x) \neq f(x+r) - f(r)) \vee (\varphi(y) \neq f(r) - f(-y+r)) \vee (\varphi(x+y) \neq f(x+r) - f(-y+r))\,]\,.$$

Finally, upper bound the probability of each of the 3 events by $2\rho < 1/3$.

Exercise 2 — **10** points

Let R_n denote a totally ordered set and let $[m]^\ell$ denote the hypergrid. Prove that a function $f : [m]^\ell \to R_n$ is monotone if and only if it is monotone in each direction (that is, show that f is monotone if and only if for every $i \in [\ell]$, for every $\alpha \in [m]^{i-1}$, and for every $\beta \in [m]^{\ell-i}$, the function $f_i(z) := f(\alpha z \beta)$ is monotone in z).

Exercise 3 — **10** points

Recall the "edge test" algorithm from the lecture. Further, let $f : \{0,1\}^\ell \to \{0,1\}$ denote a function, let $\delta_{\mathtt{M}}(f)$ denote the relative distance of f from being monotone, and let $\rho(f)$ denote the probability that the "edge test" rejects f. Note that we have $\rho(f) \le 2\delta_{\mathtt{M}}(f)$ for every f.

Show that for $f(x) = \mathrm{wt}(x) \bmod 2$, we have $\delta_{\mathtt{M}}(f) \approx 0.5$ and $\rho(f) \approx 0.5$.

Exercise 4 — **10** points

Let P denote a poset with n elements, let R denote an ordered set, and let $f : P \to R$ denote a function that is at distance δ from the set of monotone functions over P (with range R). Consider the graph G_f with consisting of vertices for each element of P and edges $\{x, y\}$ for each $x, y \in P$ that satisfy $x <_P y$, but not $f(x) >_R f(y)$.

Prove that any vertex cover of the graph G_f contains at least δn vertices.