Themistoklis Gouleakis **Winter 2021**

Property Testing, Exercise Sheet 4

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter20/property-testing

Total Points: 40 Due: Wednesday, **January 20**, 2021

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, ***using your own words****. Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of all points on exercise sheets to be admitted to the exam.*
Please send your solutions directly to Philip (`wellnitz@mpi-inf.mpg.de`*).*

Exercise 1 — 5 + 5 points

Let $G = (V, E)$ denote a graph and let $A, B \subseteq V$ denote nonempty sets of vertices.

Recall that the *edge density* $d(A, B)$ is defined as $d(A, B) := |E(A, B)|/|A|\cdot|B|$. Further, for a $\gamma \in (0, 1)$, we say that the pair (A, B) is *γ-regular* if for every $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq \gamma \cdot |A|$ and $|B'| \geq \gamma \cdot |B|$ we have that $|d(A', B') - d(A, B)| \leq \gamma$.

Now, suppose that (A, B) is a γ-regular pair of edge density ρ, and let $N_B(v) := \{u \in B \mid \{u, v\} \in E\}$ denote the neighbors of a vertex $v \in A$ in the set B. Prove the following claims:

a) At least $(1 - 2\gamma) \cdot |A|$ vertices $v \in A$ satisfy

$$(\rho - \gamma) \cdot |B| \leq |N_B(v)| \leq (\rho + \gamma) \cdot |B|.$$

b) If $\rho \geq 2\gamma$, then at least a $(1 - 2\gamma)^2$-fraction of the vertex pairs $(v_1, v_2) \in A^2$ satisfy

$$(\rho^2 - 2\gamma) \cdot |B| \leq |N_B(v_1) \cap N_B(v_2)| \leq (\rho^2 + 2\gamma) \cdot |B|.$$

Exercise 2 — 10 points

Let A and B denote (disjoint) sets of size n each. Prove that for a random bipartite graph between A and B, the pair (A, B) is γ-regular with probability at least $1 - \exp(-\gamma^4 - n^2 + 2n)$.

Exercise 3 — 10 points

Show that there are (bounded degree) n-vertex graphs that are $\Omega(1)$-far from being bipartite but have no odd-cycle of length $o(\log k)$.

Exercise 4 — 10 points

Let $G = (V, E)$ denote a graph and uniformly select a vertex $s \in V$. Further, assume that G is rapidly-mixing (for ℓ-step lazy random walks starting at s). Let $p_1^j(u)$ denote the probability that a lazy random walk of length j that starts at s reaches u while making an odd number of real (that is, non-self-loop) steps.

Prove that $p_1^{\ell-1}(u) \in [0.9 \cdot p_1^{\ell}(u), 1.1 \cdot p_1^{\ell}(u)]$.