Philip Wellnitz, Barış Can Esmer **Winter 2022/23**

— Randomized Algorithms and Probabilistic Analysis of Algorithms, Exercise Sheet 2 —

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter22/random

Total Points: 100 Due: Wednesday, **November 23**, 2022

You are allowed to collaborate on the exercise sheets. Justify your answers. Cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of all points on exercise sheets to be admitted to the exam.
Please hand in your solutions before the lecture on the day of the deadline.

Exercise 1 — **5 + 5 + 5 + 10** points

Let X denote a binomial random variable with parameters n and p, that is, for $0 \leq j \leq n$, we have $\mathsf{Pr}[\, X = j\,] = \binom{n}{j} p^j (1-p)^{n-j}$. Prove the following.

a) $\sum_{j=0}^{n} \mathsf{Pr}[\, X = j\,] = 1$.

b) $\mathbb{E}[\, X\,] = np$.

c) $\mathsf{Var}[\, X\,] = np(1-p)$.

d) $M_X(t) = (1 - p + p\,\mathrm{e}^t)^n$.

Exercise 2 — **25** points

Generalize the randomized algorithm for computing the median of n distinct numbers from the lecture so that it computes the k-th largest element instead, maintaining both running time and success probability.

Exercise 3 — **25** points

Prove the following generalization of Chebyshev's Inequality.

Write X for a random variable and k for a (positive) even integer. Suppose that $\mathbb{E}\left[\, (X - \mathbb{E}[\, X\,])^k\,\right]$ is finite. Show that

$$\mathsf{Pr}\left[\, |X - \mathbb{E}[\, X\,]\,| > t\sqrt[k]{\mathbb{E}[\, (X - \mathbb{E}[\, X\,])^k\,]}\,\right] \leq t^{-k}.$$

Exercise 4 — **25** points

Suppose you have oracle access to a random number generator that generates integers from $1, \dots, n$ uniformly at random, and suppose that this oracle is your only source of randomness. Using said oracle, show how to sample a random permutation of $1, \dots, n$ uniformly at random.