Philip Wellnitz, Barış Can Esmer **Winter 2022/23**

— Randomized Algorithms and Probabilistic Analysis of Algorithms, Exercise Sheet 6 —

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter22/random

Total Points: 100 Due: Wednesday, **January 25**, 2023

You are allowed to collaborate on the exercise sheets. Justify your answers. Cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of all points on exercise sheets to be admitted to the exam. Please hand in your solutions before the lecture on the day of the deadline.

Exercise 1 — **25** points

Write $G = (V, E)$ for a connected, undirected (non-bipartite) graph with maximum degree $\Delta := \max_{v\in V} |N(v)|$. Prove that for any distribution $\pi : V \to [0, 1], \sum_{v\in V} \pi_v = 1$, and any $M \geq \delta$, the following Markov chain on V has π as its stationary distribution:

$$P_{u,v} := \begin{cases} (1/M)\min\{1, \pi_v/\pi_u\}, & \text{if } u \neq v \text{ and } \{u, v\} \in E, \\ 0, & \text{if } u \neq v \text{ and } \{u, v\} \notin E, \\ 1 - \sum_{w\neq u} P_{u,w}, & \text{if } u = v. \end{cases}$$

Exercise 2 — **10 + 5 + 10** points

One may shuffle a deck of n cards by repeatedly picking a card uniformly and independently at random and putting the chosen card on top of the deck. Thinking of all permutations of the cards as the states, we can view this process as a Markov chain, which we call the card-shuffling chain.

a) Show that the card-shuffling chain is finite, irreducible, and aperiodic and compute its stationary distribution.

Consider the following coupling $Z_t = (X_t, Y_t)$ for the card-shuffling chain. To obtain X_{t+1} from X_t, we proceed as before, that is, we pick a card C uniformly and independently at random from among the n cards and move it to the top. To obtain Y_{t+1} from Y_t, we pick the card with value C and place it at the top of the chain.

b) Prove that Z_t is indeed a valid coupling for the card-shuffling chain.

c) Using the coupling Z_t, prove that the card-shuffling chain is rapidly mixing, that is, its mixing time is polynomial in $\log(1/\varepsilon)$ and n. *(Hint: Recall the Coupon Collector's Problem.)*

Exercise 3 — **10 + 10** points

a) Prove that any sequence $Z_0, Z_1, \ldots, Z_n$ that is a martingale with respect to $X_0, X_1, \ldots, X_n$ is also a martingale with respect to itself.

b) For a sequence of random variables $X_0, X_1, \ldots$, write $S_i := \sum_{j=0}^{i} X_i$. Show that if $S_0, S_1, \ldots$ is a martingale with respect to $X_0, X_1, \ldots$, then for all $i \neq j$, we have $\mathbb{E}[\, X_i X_j \,] = 0$.

Exercise 4 — **30** points

Prove the following generalization of the Azuma-Hoeffding inequality seen in class.

Theorem. *For any martingale $X_0, \cdots X_n$, constants d_k and random variables B_k (that may be functions in $X_0, \ldots, X_{k-1}$) that satisfy*

$$B_k \leq X_k - X_{k-1} \leq B_k + d_k,$$

we have for all $t \geq 0$ and any $\varepsilon > 0$ that

$$\Pr[\, |X_t - X_0| \geq \varepsilon \,] \leq 2e^{-2\varepsilon^2/(\sum_{k=1}^{t} d_k^2)}.$$