Probabilistic Graphical Models
Exercise Sheet No. 4

Due Date: December 14th, 10 am

Hand in: By e-mail to (jlange[at]mpi-inf.mpg.de). Your submission must include your source code files. Begin the subject of your e-mail with [pgm].

1 Submodularity

Points: 9

Recall from the previous exercise sheet the energy function

$$E_{h,\beta,\eta}(x,y) = h\sum_{i=1}^{N} x_i - \beta\sum_{i\sim j} x_i x_j - \eta\sum_{i=1}^{N} x_i y_i, \tag{1}$$

where $x, y \in \{-1,1\}^N$ and $\beta, \eta \geq 0, h \in \mathbb{R}$.

a) Show that $E_{h,\beta,\eta}$ as a function of x is submodular for all values of y.

Let $E_\theta(x) = \sum_{i\in V} \theta_i(x_i) + \sum_{ij\in E} \theta_{ij}(x_i, x_j)$ with $x_i \in \mathcal{X}_i$ be a pairwise energy function. Suppose $\mathcal{X}_i = \{0,1\}$ for all $i \in V$.

b) Find a reparametrization θ^φ and some $\alpha_{ij} \in \mathbb{R}$ such that the pairwise potential θ^φ_{ij} can be written as

$$\theta^\varphi_{ij}(x_i, x_j) = \begin{cases} \alpha_{ij} & \text{if } x_i \neq x_j \\ 0 & \text{else.} \end{cases}$$

Now suppose $\mathcal{X}_i = \{1, \dots, n\}$ for all $i \in V$. The definition of submodularity from the lecture is extended to multi-label energy functions by extending the definition of submodular potentials as follows. The pairwise potential θ_{ij} is called *submodular* if for all $x_i, x'_i \in \mathcal{X}_i$ with $x_i \leq x'_i$ and all $x_j, x'_j \in \mathcal{X}_j$ with $x_j \leq x'_j$ it holds that

$$\theta_{ij}(x'_i, x_j) + \theta_{ij}(x_i, x'_j) \geq \theta_{ij}(x_i, x_j) + \theta_{ij}(x'_i, x'_j). \tag{2}$$

c) Prove that the pairwise potential θ_{ij} is submodular if and only if for all $x_i \in \mathcal{X}_i$ and all $x_j \in \mathcal{X}_j$ it holds that

$$\theta_{ij}(x_i, x_j) + \theta_{ij}(x_i+1, x_j+1) - \theta_{ij}(x_i+1, x_j) - \theta_{ij}(x_i, x_j+1) \leq 0.$$

2 Sequential Tree-reweighted Message Passing

Points: 7

a) Implement Sequential Tree-reweighted Message Passing (TRWS) to solve the MAP-inference problem from the previous exercise sheet.

b) Reconstruct an image x by TRWS with primal rounding and subsequent application of ICM. Compare the energy and loss $L(x, x^{\mathrm{GT}})$ to the MPLP solution from the previous exercise sheet.

c) We define one outer iteration of MPLP as the update of all edges. Further, we define one outer iteration of TRWS as one total backward-forward pass. Compare the number of outer iterations of MPLP to the number of outer iterations of TRWS when solving the dual problem to a given accuracy.