Exercise 1 - Spectrum of Symmetric Matrices

Any real, symmetric matrix $A \in \mathbb{R}^{n \times n}$ has the decomposition

$$A = U \Sigma U^T,$$

where $\Sigma \in \mathbb{R}^{n \times n}$ is a diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ on the diagonal and $U$ is an orthogonal matrix in $\mathbb{R}^{n \times n}$, that is $UU^T = U^TU = I$, which contains the corresponding eigenvectors (more precisely: an orthogonal basis of the eigenspace of the corresponding eigenvalue).

a. Derive the eigenvalues and eigenvectors of $A_k$ (matrix product with itself) for $k \in \mathbb{N}$.

b. Prove that

$$\langle x, Ax \rangle \langle x, x \rangle \leq \lambda_{\text{max}}(A),$$

where $\langle x, y \rangle = x^T y$ is the inner product in $\mathbb{R}^n$, $\lambda_{\text{max}}(A)$ is the largest eigenvalue of $A$.

Exercise 2 - Empirical Mean and Covariance

Given a set of $n$ points $X = [x_1, \ldots, x_n]$, where $x_n \in \mathbb{R}^d$, $X \in \mathbb{R}^{d \times n}$.

a. Derive the minimizer $c^*$ for function

$$f(c) = \sum_{i=1}^n \|x_i - c\|_2^2,$$

where $\|x\|_2$ denotes the Euclidean norm $\|c\|_2 = \sqrt{\sum_{j=1}^d c_j^2}$.

b. Show that the empirical covariance matrix for $X$

$$\Sigma_X = \frac{1}{n} \sum_{i=1}^n (x_i - \mu)(x_i - \mu)^T$$

is positive semi-definite, that is $w^T \Sigma_X w \geq 0$, for all $w \in \mathbb{R}^d$, $\mu = \frac{1}{n} \sum_{i=1}^n x_i$.

Hint: consider using the Cauchy-Schwarz inequality, $\langle u, v \rangle^2 \leq \|u\|^2 \|v\|^2$.

Exercise 3 - Multivariate Gaussian

In the lecture we have seen the multivariate Gaussian $x \sim N(\mu, \Sigma)$ where the density function is defined as

$$f(x) = (2\pi)^{-\frac{d}{2}} |\Sigma|^{-\frac{1}{2}} \exp(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu))$$

Now we have $n$ multivariate Gaussian random vectors $\{x_i\}_{i=1}^n$, where $x_i \sim N(0, \Sigma_i), x_i \in \mathbb{R}^d$.

a. Consider the case where all the random vectors are mutually independent, derive the density function for $\sum_{i=1}^n x_i$.

b. Consider the case $n = 2$. Given the covariance matrix $\text{cov}(x_1, x_2) = C$, derive the density function for the joint vector $(x_1, x_2) \in \mathbb{R}^{2d}$.