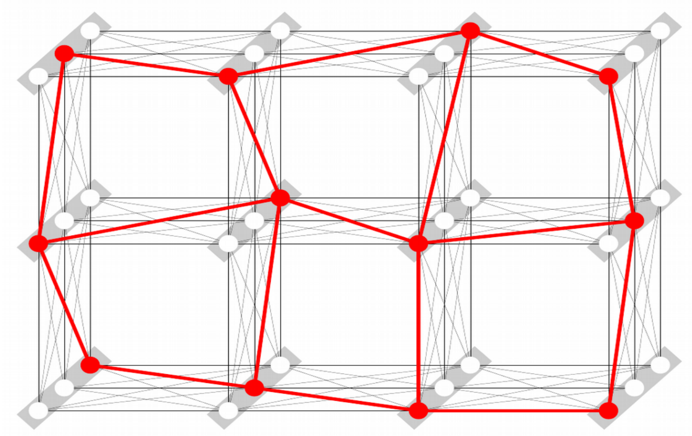
Combinatorial Solution Set

$$x^* = \min_{\overline{x} \in X_V} E_\theta(\overline{x}) := \min_{\overline{x} \in X_V} \sum_{v \in V} \theta_v(x_v) + \sum_{uv \in \mathcal{E}} \theta_{uv}(x_u, x_v)$$



Number of possible labelings: $\prod_{v \in V} |X_V|$

Example: 2x2 grid and full graphs with 10 labels

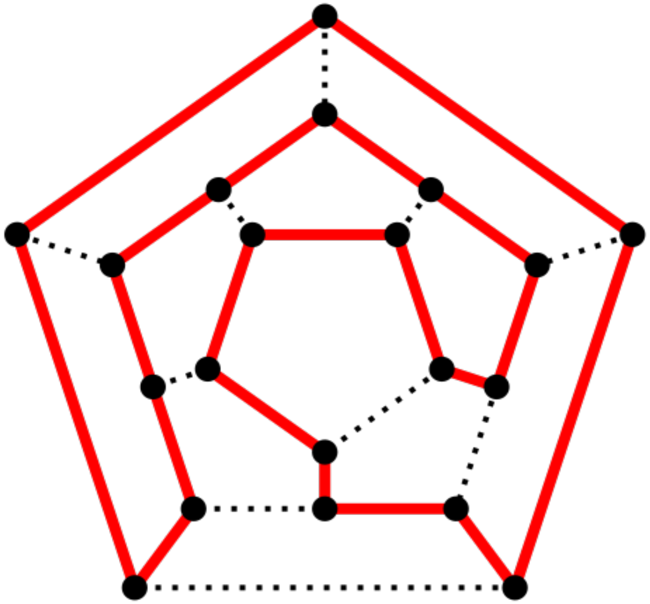
1) 400
2) 440
3) 40
4) 1000
5) 10000
6) No correct answer

Picture: T Werner. A Linear Programming Approach to Max-sum Problem: A Review

Combinatorial Solution Set = No Polynomial Algorithm?

Hamiltonian Cycle

A **Hamiltonian cycle** in a undirected graph is the cycle that visits each vertex exactly once.



Determining whether such a cycle exists in a graph is NP-complete.

Picture: Wikipedia

Hamiltonian Cycle Reduces to MAP-Inference

Graph $(V, \mathcal{E})$ for a Hamiltonian cycle

Full graph $(V, V \times V)$ for the MAP-inference problem

The set of labels $X_v = V$ for all $v \in V$.

What about an inverse reduction:
MAP-inference to a Hamiltonian cycle?

How to Deal With NP-Hard Problems?

1) Identify polynomially solvable subclasses

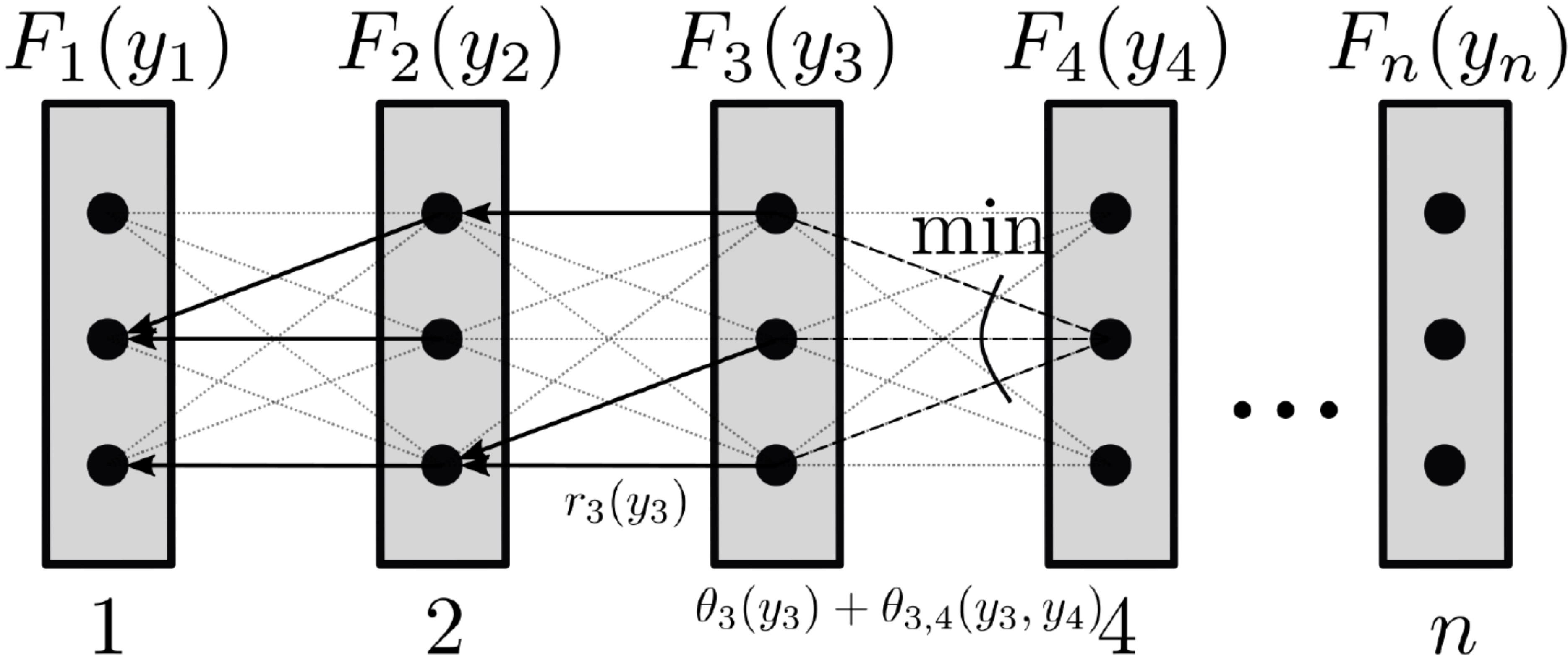
2) Consider its convex relaxations (polynomially solvable approximations)

3) Develop approximate algorithms

We start with some background...

Acyclic models: chains

$$F_i(s) := \min_{t \in \mathcal{Y}_{i-1}} \left(F_{i-1}(t) + \theta_{i-1}(t) + \theta_{i-1,i}(t,s) \right)$$



Acyclic models: recursive computation

$$\min_{y \in \mathcal{Y}_\mathcal{V}} E(y;\theta) = \min_{y_1,\dots,y_n} \left(\sum_{i=1}^{n-1} \Big(\theta_i(y_i) + \theta_{i,i+1}(y_i, y_{i+1}) \Big) + \theta_n(y_n) \right)$$

$$= \min_{y_2,\dots,y_n} \left(\underbrace{\min_{y_1 \in \mathcal{Y}_1} \Big(\theta_1(y_1) + \theta_{1,2}(y_1, y_2) \Big)}_{F_2(y_2)} \right.$$

$$\left. + \sum_{i=2}^{n-1} \Big(\theta_i(y_i) + \theta_{i,i+1}(y_i, y_{i+1}) \Big) + \theta_n(y_n) \right)$$

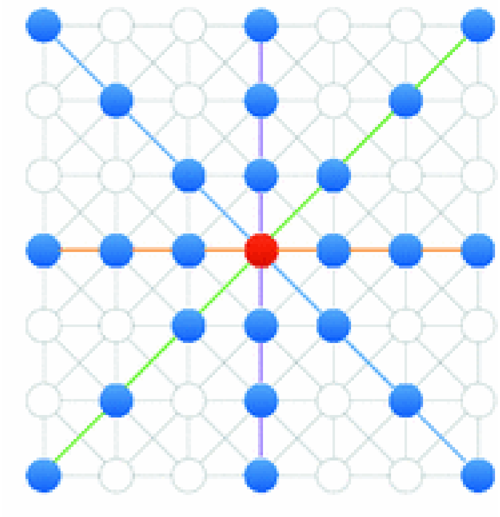
$$= \min_{y_3,\dots,y_n} \left(\underbrace{\min_{y_2 \in \mathcal{Y}_2} \Big(F_2(y_2) + \theta_2(y_2) + \theta_{2,3}(y_2, y_3) \Big)}_{F_3(y_3)} \right.$$

$$\left. + \sum_{i=3}^{n-1} \Big(\theta_i(y_i) + \theta_{i,i+1}(y_i, y_{i+1}) \Big) + \theta_n(y_n) \right)$$

$$\dots$$

$$= \min_{y_n \in \mathcal{Y}_n} \Big(F_n(y_n) + \theta_n(y_n) \Big).$$

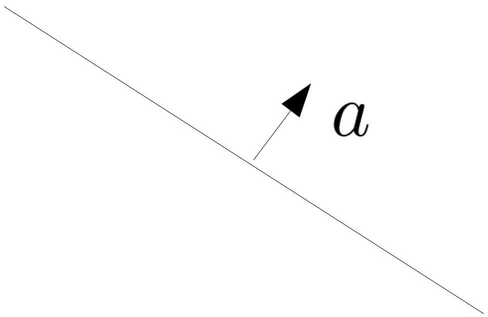
Semiglobal matching



Linear Programs and Their Geometry

Integer linear programs is a standard way to describe combinatorial problems.

Hyperplane: $\{x \in \mathbb{R}^n \colon \langle a, x \rangle = b\}$



Half-space $\{x \in \mathbb{R}^n \colon \langle a, x \rangle \leq b\}$

Polyhedron: $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$

Polytope:

Linear Programs and Their Geometry

Polyhedron: $Ax \leq b$

Polyhedron?
$$Ax = b$$
$$x_i \geq 0,\ i \in I \subset \{1, \ldots, n\}$$

1) Yes
2) No
3) Don't know

Linear Programs and Their Geometry

Polyhedron: $Ax \leq b$

Polyhedron?
$$Ax = b$$
$$x_i \geq 0, \; i \in I \subset \{1, \ldots, n\}$$

1) Yes
2) No
3) Don't know

$$\mathrm{Ax} \leq b$$
$$-Ax \leq -b$$
$$-x_i \leq 0, i \in I$$

What about an inverse transformation:

$$Ax \leq b \quad \text{to} \quad \begin{aligned} &Ax = b \\ &x_i \geq 0,\ i \in I \subset \{1, \dots, n\} \end{aligned} \quad ?$$

Linear Programs and Their Geometry

What about an inverse transformation:

$$Ax \leq b \quad \text{to} \quad \begin{aligned} &Ax = b \\ &x_i \geq 0,\ i \in I \subset \{1, \ldots, n\} \end{aligned} \quad ?$$

$$\begin{aligned} Ax + x' &= b \\ x' &\geq 0 \end{aligned}$$

Linear Programs and Their Geometry

Linear programs (also holds for **max**):

Standard form

$$\min_{x \in \mathbb{R}^n} \langle c, x \rangle$$

$$Ax \leq b$$

Canonical form

$$\min_{x \in \mathbb{R}^n} \langle c, x \rangle$$

$$Ax = b$$

$$x_i \geq 0, \;\; i \in I \subset \{1, \dots, n\}$$

Vertexes and solutions:

Linear Programs and Their Geometry

Linear programs (also holds for **max**):

Standard form

$$\min_{x \in \mathbb{R}^n} \langle c, x \rangle$$

$$Ax \leq b$$

Canonical form

$$\min_{x \in \mathbb{R}^n} \langle c, x \rangle$$

$$Ax = b$$

$$x_i \geq 0, \;\; i \in I \subset \{1, \dots, n\}$$

Vertexes and solutions:

$x' \in \mathbb{R}^n$ is a vertex of a polyhedron P if there exists $c \in \mathbb{R}^n$ such that

$$\max_{x \in P} \langle c, x \rangle$$

is finite and has a unique solution

N-Dimensional Simplex

$$\Delta^n = \{p \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1\}$$

Examples: 1) Yes 2) No

0.3; 0.1; 0.7; 0.0;

N-Dimensional Simplex

$$\Delta^n = \{p \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1\}$$

Examples: 1) Yes 2) No

0.3; 0.7; 0.0; 0.0;

N-Dimensional Simplex

$$\Delta^n = \{p \in \mathbb{R}_+^n : \sum_{i=1}^{n} p_i = 1\}$$

Examples: 1) Yes 2) No

- 0.3; 0.7; 0.0; 0.0;

N-Dimensional Simplex

$$\Delta^n = \{p \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1\}$$

Examples: 1) Yes 2) No

0.0; 0.1; 0.2; 0.7;

N-Dimensional Simplex

$$\Delta^n = \{p \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1\}$$

Examples: 1) Yes 2) No

0.0; 0.0; 0.0; 1.0;

N-Dimensional Simplex

$$\Delta^n = \{p \in \mathbb{R}_+^n : \sum_{i=1}^{n} p_i = 1\}$$

Examples: 1) Yes 2) No

0.0; 0.0; 1.0; 0.0;

N-Dimensional Simplex

$$\Delta^n = \{p \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1\}$$

Examples: 1) Yes 2) No

0.0; 0.0; 0.0; 0.0;

N-Dimensional Simplex

$$\Delta^n = \{p \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1\}$$

Draw your own picture:

Convex Hull

$$\delta^i \in \mathbb{R}^N, \; i = 1, \ldots, n$$

- arbitrary vectors

Convex hull: **Always a polytope!**

$$\operatorname{conv}\{\delta^i : i = 1, \ldots, n\} := \{s : \; s = \sum_{i=1}^{n} p_i \delta^i, \; p \in \Delta^n\}$$

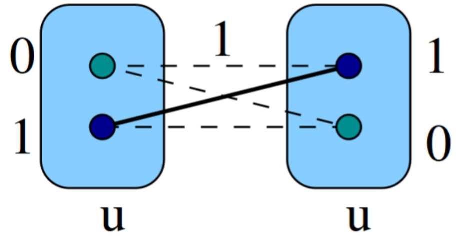
Convex Hull

Lemma 1:

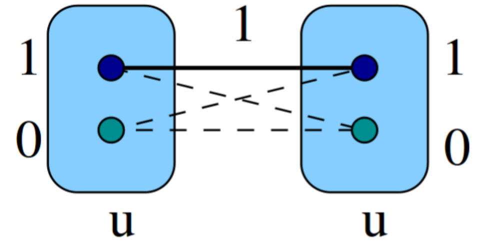
$$\min_{i=1,\ldots,n} \{a_i\} = \min_{p\in\Delta^n} \sum_{i=1}^{n} p_i a_i = \min_{\mu\in\mathrm{conv}\{a_i,\ i=1,\ldots,n\}} \mu$$

Marginal Polytope: Labeling → Vector

Mapping: labeling → binary vector $\delta : X_V \to \{0,1\}^I$



$$\delta(x^1) = \underbrace{01}_{u}\ \underbrace{0100}_{uv}\ \underbrace{10}_{v}$$

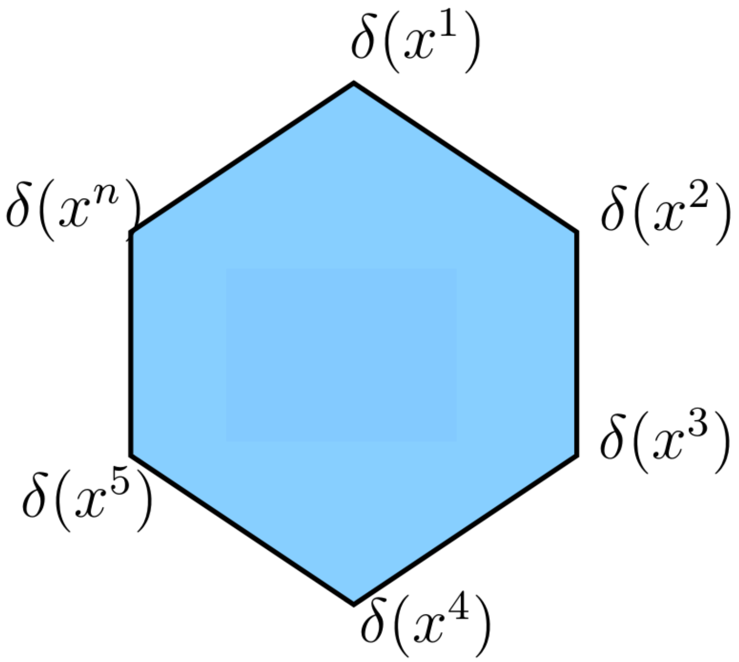


$$\delta(x^2) = \underbrace{10}_{u}\ \underbrace{1000}_{uv}\ \underbrace{10}_{v}$$

$$\min_{x \in X_V} \sum_{v \in V} \theta_v(x_v) + \sum_{uv \in \mathcal{E}} \theta_{uv}(x_u, x_v) \equiv \min_{x \in X_V} \langle \theta, \delta(x) \rangle$$

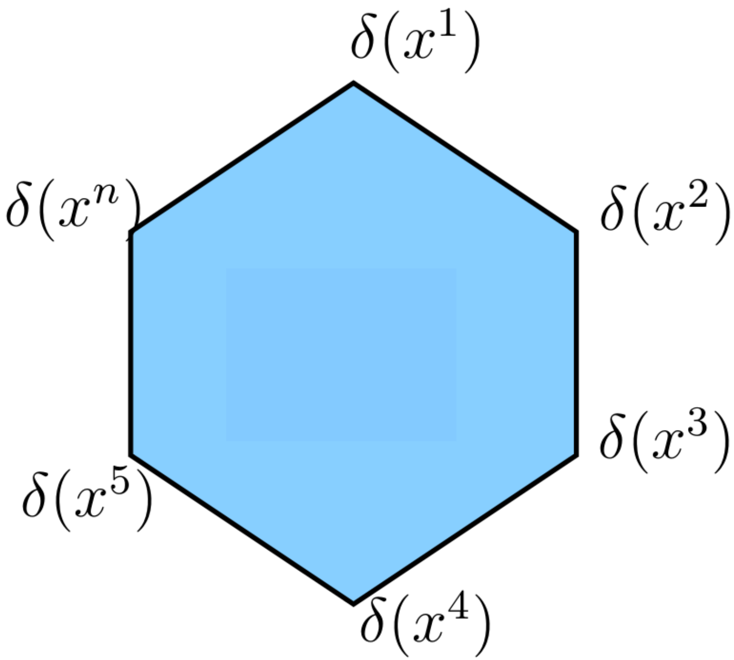
Marginal Polytope

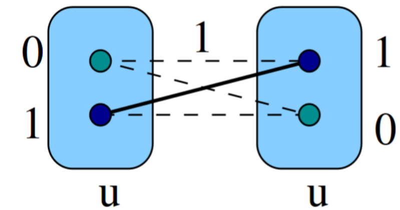
Marginal polytope: $M := \mathrm{conv}\{\delta(x) \in \mathbb{R}^I : x \in X_V\}$



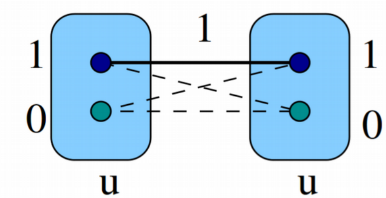
Marginal Polytope

Marginal polytope: $M := \operatorname{conv}\{\delta(x) \in \mathbb{R}^I : x \in X_V\}$



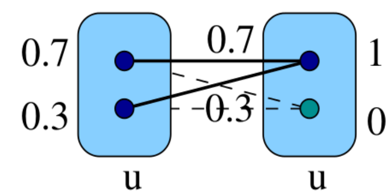


$$\delta(x^1) = \underbrace{01}_{u}\ \underbrace{0100}_{uv}\ \underbrace{10}_{v}$$



$$\delta(x^2) = \underbrace{10}_{u}\ \underbrace{1000}_{uv}\ \underbrace{10}_{v}$$

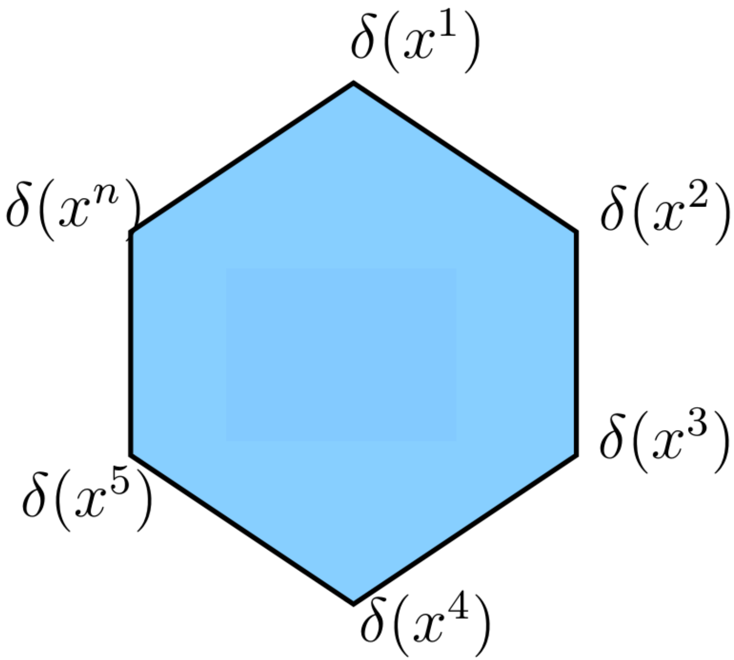
$0.3x^1 + 0.7x^2$:



$$0.3\delta(x^1) + 0.7\delta(x^2) = \underbrace{0.7\,0.3}_{u}\ \underbrace{0.7\,0.3\,00}_{uv}\ \underbrace{10}_{v}$$

Marginal Polytope

Marginal polytope: $$M := \mathrm{conv}\{\delta(x) \in \mathbb{R}^I : x \in X_V\}$$

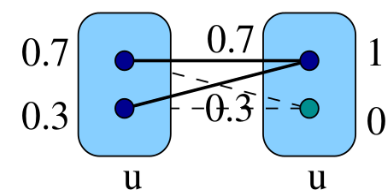


Corollary 1:

$$\mu_v \in \Delta^{|X_v|} \text{ for any } v \in V$$

$$\mu_{uv} \in \Delta^{|X_{uv}|} \text{ for any } uv \in \mathcal{E}$$

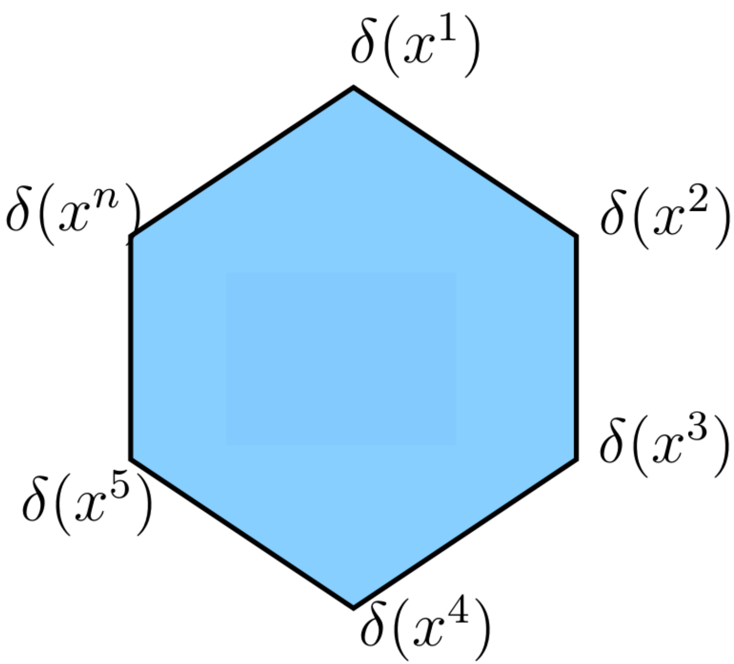
$0.3x^1 + 0.7x^2$:



$$0.3\delta(x^1) + 0.7\delta(x^2) = \underbrace{0.7\;0.3}_{u}\;\underbrace{0.7\;0.3\;0\;0}_{uv}\;\underbrace{1\;0}_{v}$$

Marginal Polytope

Marginal polytope: $$M := \mathrm{conv}\{\delta(x) \in \mathbb{R}^I : x \in X_V\}$$

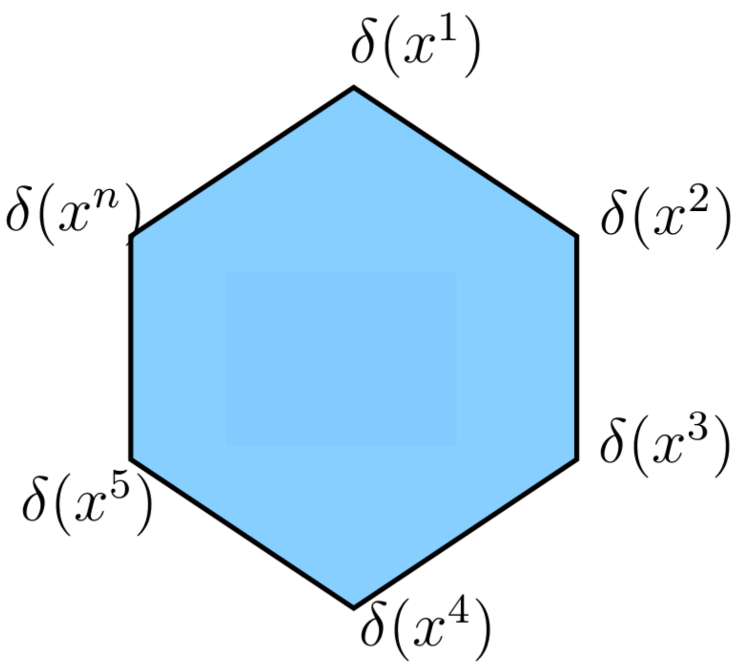


Proposition 2:

$$\min_{x \in X_V} \langle \theta, \delta(x) \rangle = \min_{\mu \in M} \langle \theta, \mu \rangle$$

Marginal Polytope

Marginal polytope: $M := \mathrm{conv}\{\delta(x) \in \mathbb{R}^I : x \in X_V\}$



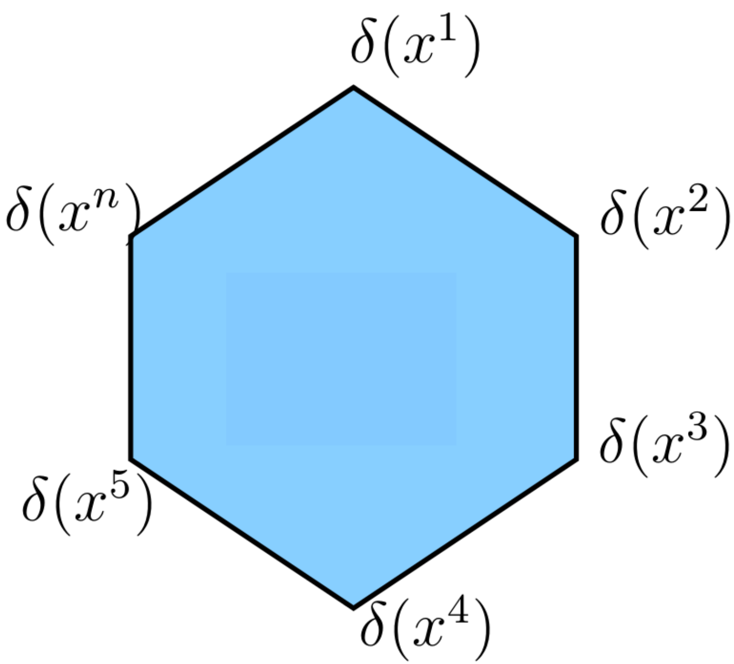
Proposition 2:

$$\min_{x \in X_V} \langle \theta, \delta(x) \rangle = \min_{\mu \in M} \langle \theta, \mu \rangle$$

NP-hard problem = linear problem?

Marginal Polytope

Marginal polytope: $M := \operatorname{conv}\{\delta(x) \in \mathbb{R}^I : x \in X_V\}$



Proposition 2:

$$\min_{x \in X_V} \langle \theta, \delta(x) \rangle = \min_{\mu \in M} \langle \theta, \mu \rangle$$

NP-hard problem = linear problem?

$$\sum_{x \in X_V} p_x \delta(x) = \mu$$

$$\sum_{x \in X_V} p_x = 1$$

$$p_x \geq 0, \quad x \in X_V$$

Yes, with an exponential number of constraints!

Local (Marginal) Polytope

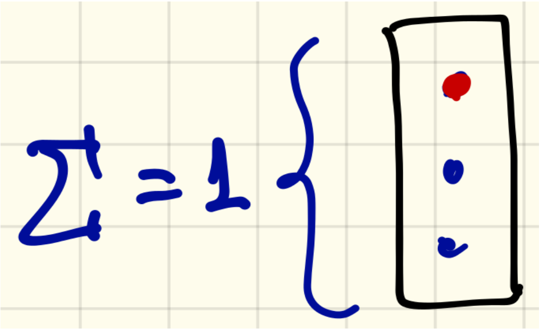
$$\cancel{p_x \geq 0, \; x \in X_V}$$ - non-local, exponential number

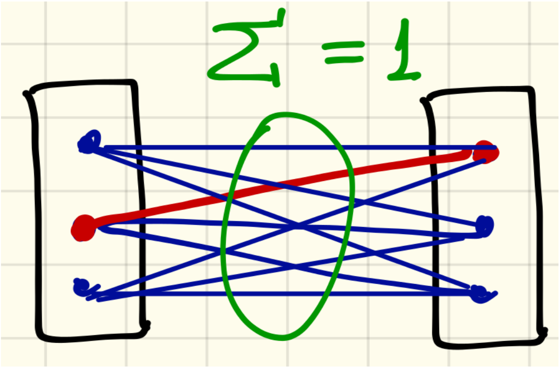
Local constraints are required to keep their number polynomial. Recall:

$$\sum_{s \in X_w} \mu_w(s) = 1, \qquad w \in V \times \mathcal{E}$$

- simplex constraints

$$\mu_w \geq 0\,.$$





Local (Marginal) Polytope

$$\cancel{p_x \geq 0,\ x \in X_V}$$

- non-local, exponential number

Local constraints are required to keep their number polynomial. Recall:

$$\sum_{s \in X_w} \mu_w(s) = 1, \qquad w \in V \times \mathcal{E}$$

- simplex constraints

$$\mu_w \geq 0\,.$$

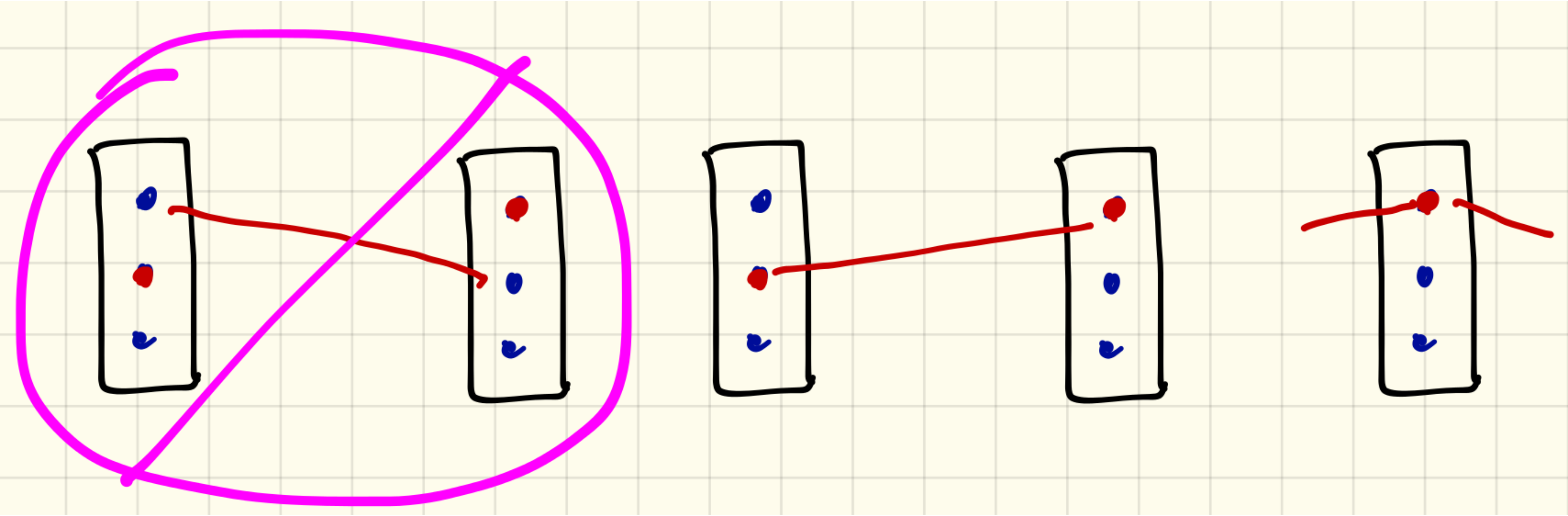
Insufficient, the inference problem separates out into independent parts for each factor:

$$\min_{\mu} \langle \theta, \mu \rangle$$
$$\text{s.t.} \quad \sum_{s \in X_w} \mu_w(s) = 1,\ w \in V \times \mathcal{E}$$
$$\mu_w \geq 0\,.$$

$$\Rightarrow$$

$$\sum_{w \in V \times \mathcal{E}} \min_{\mu_w \in X_w} \langle \theta_w, \mu_w \rangle$$
$$\text{s.t.} \sum_{s \in X_w} \mu_w(s) = 1$$
$$\mu_w \geq 0\,.$$

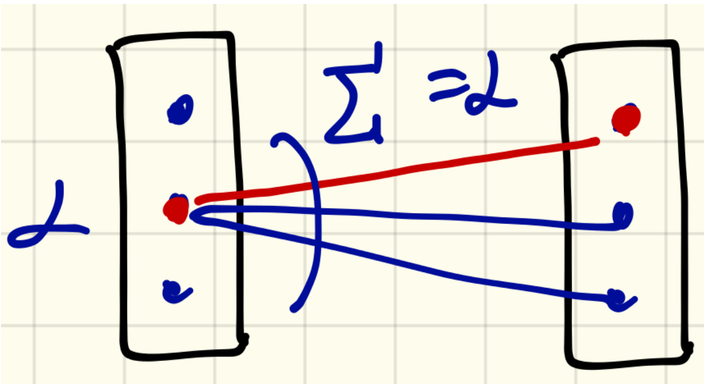
Local (Marginal) Polytope

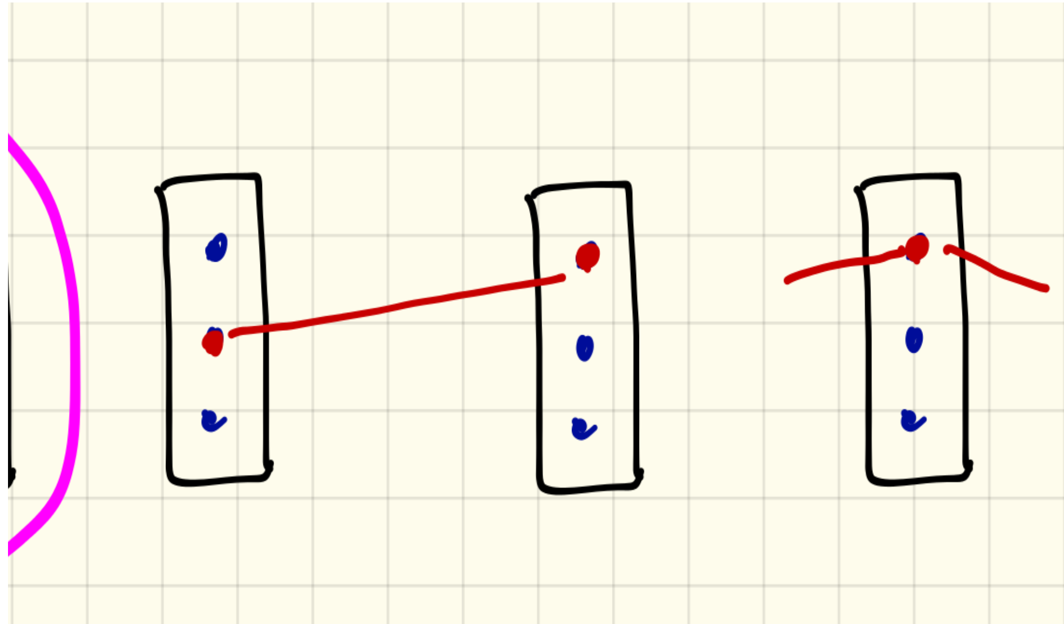


$$\mu_u(s) = 1 \;\Rightarrow\; \forall v \in N_b(u)\; \exists t \in X_v : \mu_{uv}(s,t) = 1$$

$$\mu_{uv}(s,t) = 1 \;\Rightarrow\; \mu_u(s) = 1 \text{ and } \mu_v(t) = 1 (1)$$

Local (Marginal) Polytope





$$\sum_{s \in X_u} \mu_{uv}(s,t) = \mu_v(t),\ \forall\ v \in V,\ t \in X_v$$

$$\sum_{t \in X_v} \mu_{uv}(s,t) = \mu_u(s),\ \forall\ u \in V,\ s \in X_u\ .$$

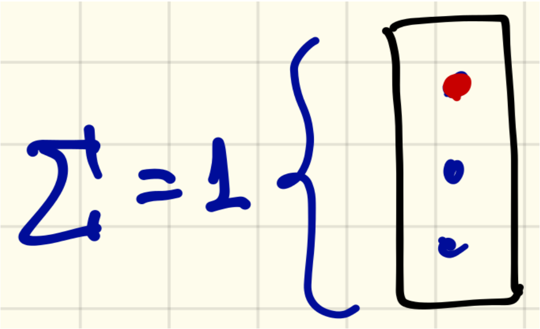
Local (Marginal) Polytope

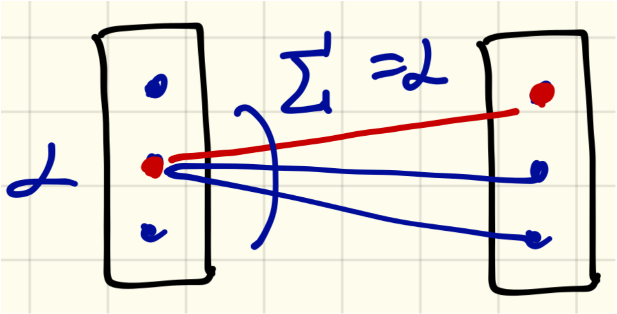
$$\mu^* = \arg\min_{\mu \in \mathcal{L}} \langle \theta, \mu \rangle$$

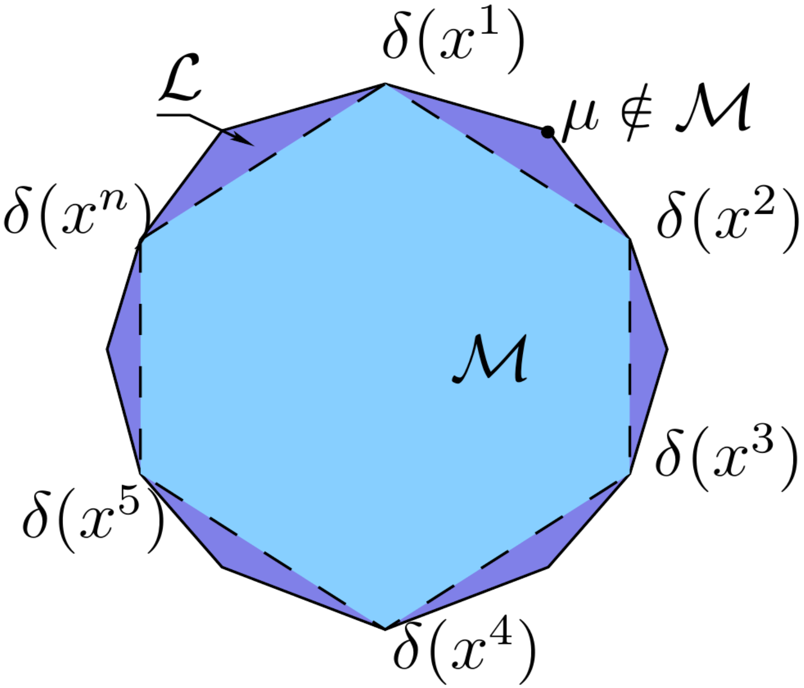
Local polytope (LP) relaxation; relaxed problem

μ - relaxed labeling $\quad \mu^*$ - relaxed solution

$$\mathcal{L} := \begin{cases} \sum_{s \in X_u} \mu_{uv}(s,t) = \mu_v(t), & v \in V,\ t \in X_v & (a) \\ \sum_{t \in X_v} \mu_{uv}(s,t) = \mu_u(s), & u \in V,\ s \in X_u & (b) \\ \sum_{t \in X_v} \mu_v(t) = 1, & v \in V & (c) \\ \mu \geq 0, & & (d) \\ \sum_{(s,t) \in X_{uv}} \mu_{uv}(s,t) = 1, & uv \in \mathcal{E},\ (s,t) \in X_{uv} & (e) \end{cases}$$







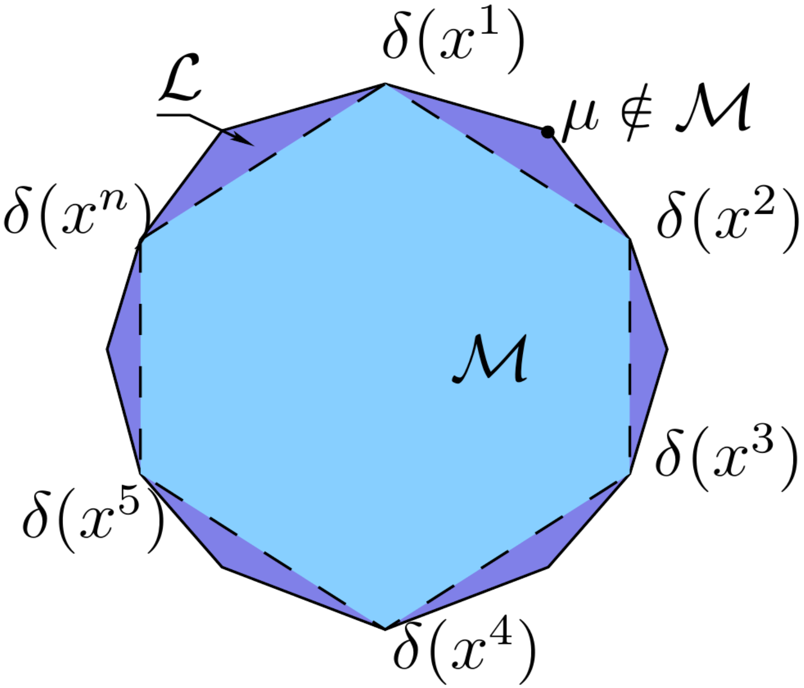
Proposition 4:

For any $x \in X_V$

$\delta(x)$ is a vertex of $\mathcal{L}$.

Proposition 5: $\mathcal{M} \subseteq \mathcal{L}$

Local (Marginal) Polytope



Proposition 4:

For any $x \in X_V$

$\delta(x)$ is a vertex of $\mathcal{L}$.

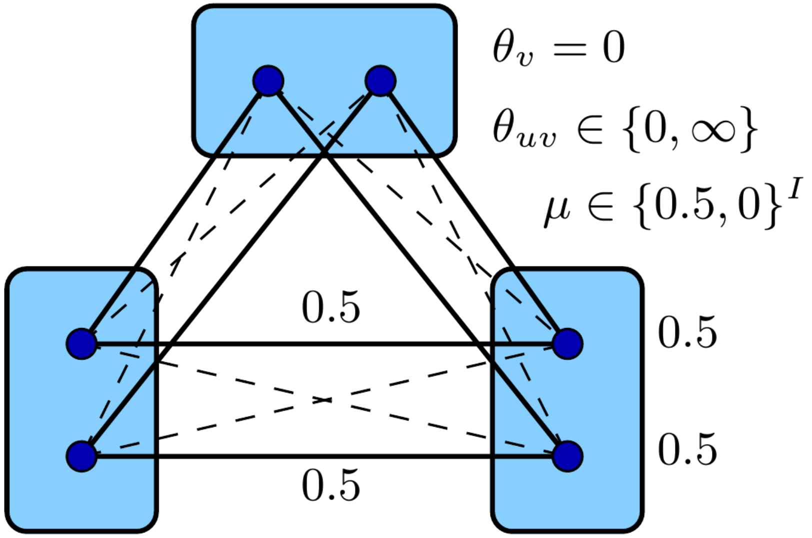
Proposition 5: $\mathcal{M} \subseteq \mathcal{L}$

Check:

$\mu^* \in \mathcal{L}$

$\mu^* \notin \mathcal{M}$

μ^* - vertex of $\mathcal{L}$



Integer Linear Programs

$$\min_{x \in \mathbb{R}^N} \langle c, x \rangle$$

$$\text{s.t. } Ax \leq b$$

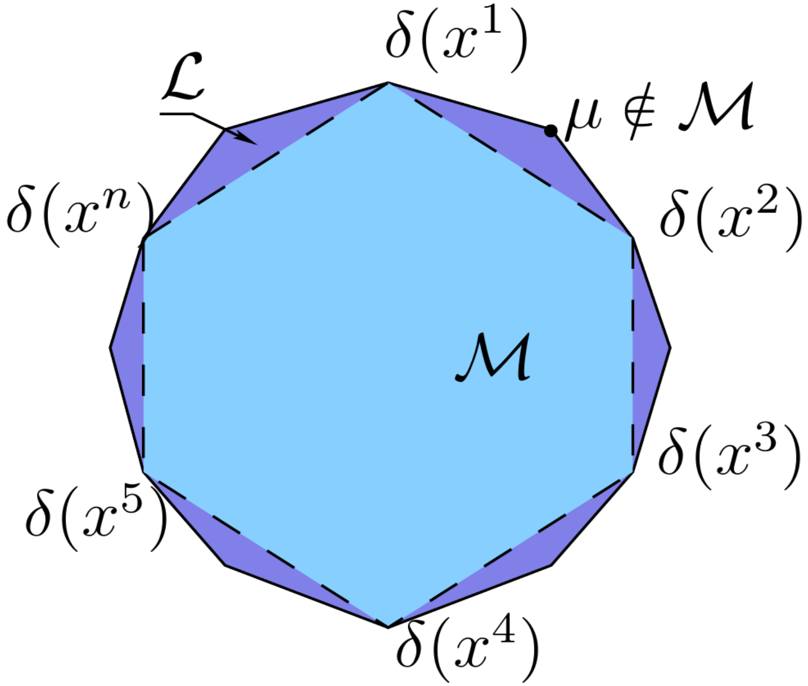
$$x_i \in \{0, 1\}, i \in I \subset \{1, \ldots, N\}$$

|*I*|=*N* – integer linear program

|*I*|<*N* – mixed integer linear program

NP-hard, standard solvers exist

Local (Marginal) Polytope



Corollary 2:

$$\min_{\mu \in \mathcal{L} \cap \{0,1\}^I} \langle \theta, \mu \rangle = \min_{x \in X_V} \langle \theta, \delta(x) \rangle$$

Local (Marginal) Polytope

Corollary 2:

$$\min_{\mu \in \mathcal{L} \cap \{0,1\}^I} \langle \theta, \mu \rangle = \min_{x \in X_V} \langle \theta, \delta(x) \rangle$$

?

1) $$\min_{\mu \in \mathcal{L}} \langle \theta, \mu \rangle \leq \min_{x \in X_V} \langle \theta, \delta(x) \rangle$$

2) $$\ldots \geq \ldots$$

3) $$\ldots = \ldots$$

4) None is correct

Local (Marginal) Polytope

Corollary 2:

$$\min_{\mu \in \mathcal{L} \cap \{0,1\}^I} \langle \theta, \mu \rangle = \min_{x \in X_V} \langle \theta, \delta(x) \rangle$$

Corollary 3:

$$\min_{\mu \in \mathcal{L}} \langle \theta, \mu \rangle \leq \min_{x \in X_V} \langle \theta, \delta(x) \rangle$$

Local Polytope: Integer/Fractional Solutions

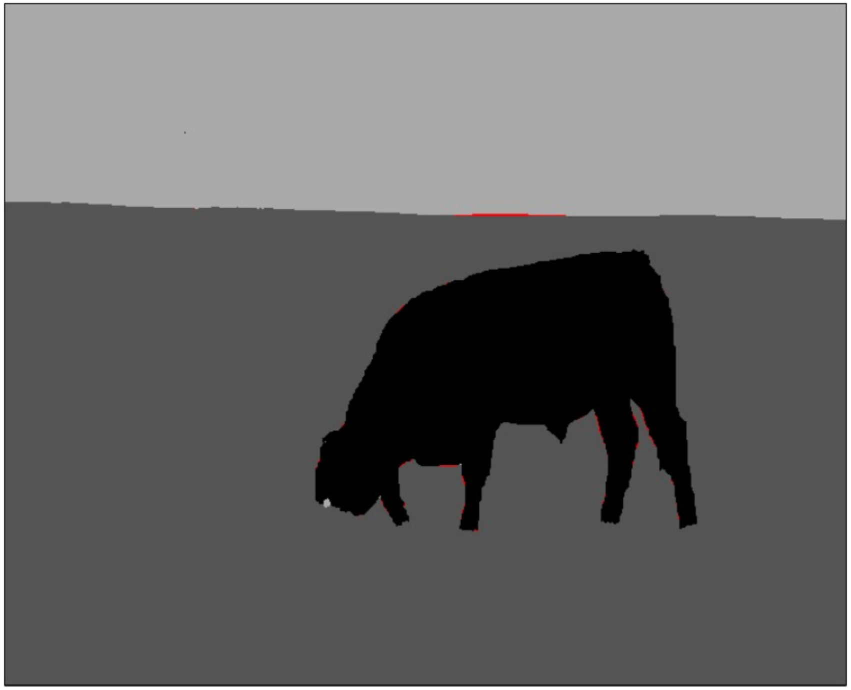
Input Stereopair

Disparities

Pictures and data: Middlebury Benchmark http://vision.middlebury.edu

Local Polytope: Integer/Fractional Solutions

Input Image



Segmentation

Pictures and data: OpenGM Benchmark http://hciweb2.iwr.uni-heidelberg.de/opengm

Local Polytope: Integer/Fractional Solutions

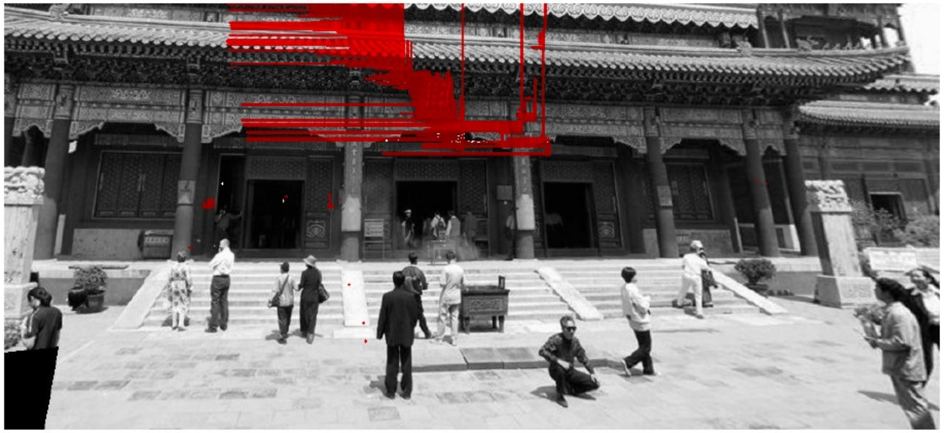
Multiple images

Stitched image

Pictures and data: Middlebury Benchmark http://vision.middlebury.edu

Local Polytope: Integer/Fractional Solutions

Multiple images



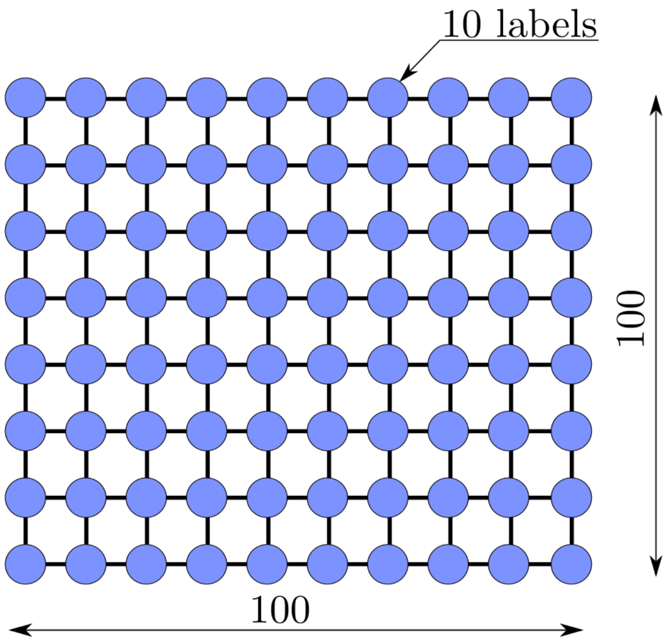
Stitched image

Pictures and data: Middlebury Benchmark http://vision.middlebury.edu

Rounding of a Relaxed Solution

$$x'_v := \arg\max_{s \in X_v} \mu'_v(s), \;\; v \in V$$

Problem Size



Pascal VOC 2012
semantic segmentation
model $\approx 500 \times 300 \times 21$ labels

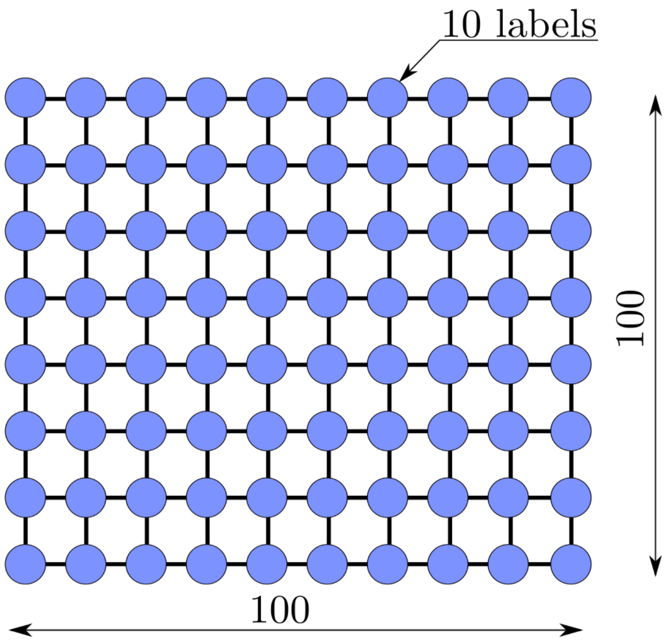
$2 \cdot 10^6$ variables

$> 10^9$ variables

Number of constraints, order of **?**

1) less
2) 1 000
3) 10 000
4) 100 000
5) 1000 000
6) more

Problem Size



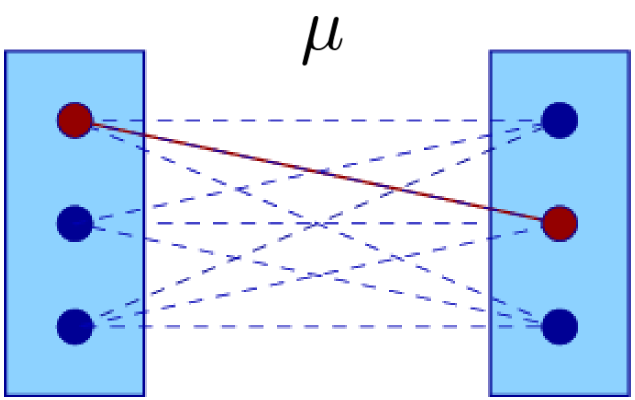
$2 \cdot 10^6$ variables

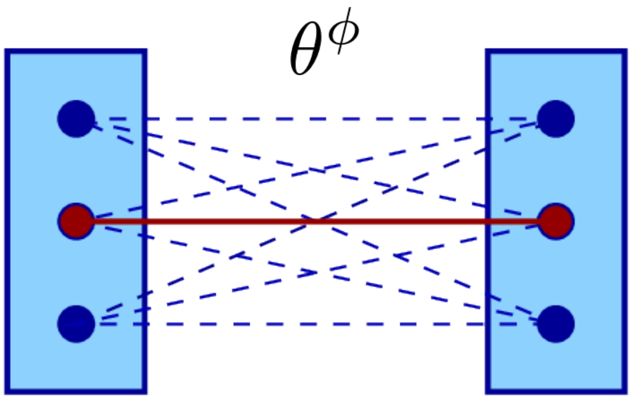
Pascal VOC 2012
semantic segmentation
model $\approx 500 \times 300 \times 21$ labels

$> 10^9$ variables

Standard (simplex, interior point) methods do not scale good enough!

Specialized solvers are needed.

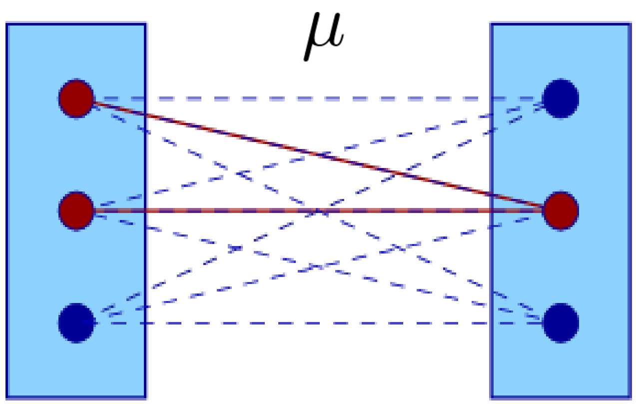


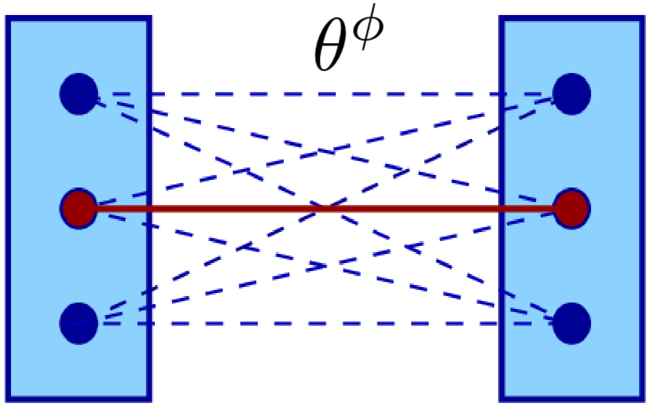


Can this primal-dual pair be optimal?

1) Yes, because the complementary slackness condition is satisfied
2) Yes, the complementary slackness condition is not satisfied, but it is not necessary for optimum
3) Yes, because μ is integer and θ_v^{ϕ}, θ_u^{ϕ} have unique minima
4) No, because the complementary slackness condition is not satisfied
5) No, because optimal μ can not be integer, since it corresponds to the relaxed problem
6) None is correct

Complementary Slackness: Example 2

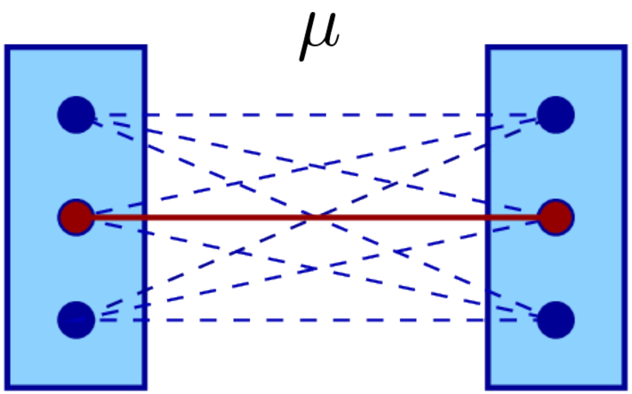


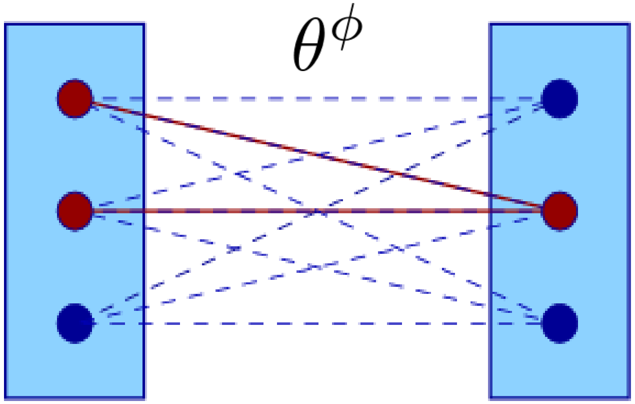


Can this primal-dual pair be optimal?

1) Yes, because the complementary slackness condition is satisfied
2) Yes, the complementary slackness condition is not satisfied, but it is not necessary for optimum
3) Yes, because μ is integer and θ_v^ϕ, θ_u^ϕ have unique minima
4) No, because the complementary slackness condition is not satisfied
5) No, because optimal μ must be integer
6) None is correct

Complementary Slackness: Example 3



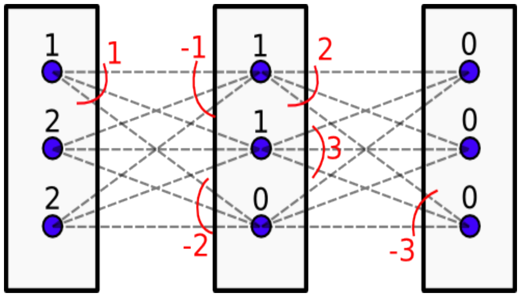


Can this primal-dual pair be optimal?

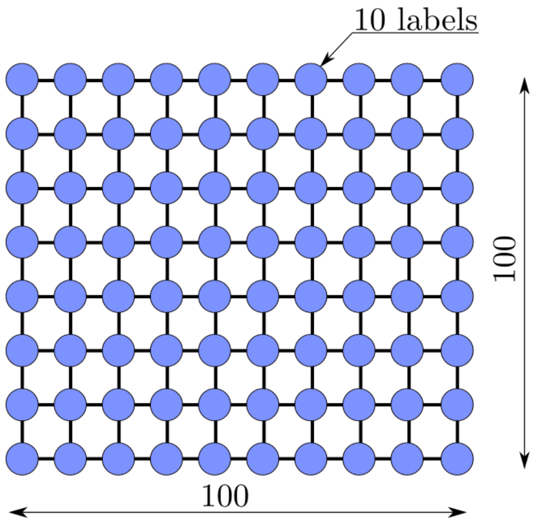
1) Yes, because the complementary slackness condition is satisfied
2) Yes, the complementary slackness condition is not satisfied, but it is not necessary for optimum
3) Yes, because μ is integer and θ_v^{ϕ}, θ_u^{ϕ} have unique minima
4) No, because the complementary slackness condition is not satisfied
5) No, because optimal θ_v^{ϕ} , θ_u^{ϕ} must be have unique minima
6) None is correct

Dual MAP LP Objective: Number of Variables

Dual: $$\max_{\phi \in \mathbb{R}^J} \sum_{v \in \mathcal{V}} \min_{s \in \mathcal{X}_v} \theta_v^\phi(s) + \sum_{uv \in \mathcal{E}} \min_{(s,t) \in \mathcal{X}_{uv}} \theta_{uv}^\phi(s,t)$$



1) 10^5 and less
2) 10^6
3) 10^7
4) 10^8
5) 10^9
6) 10^{10} and more



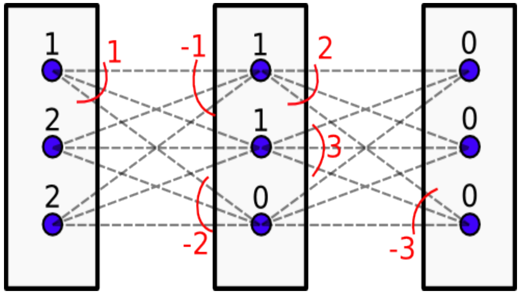
Pascal VOC

$500 \times 300 \times 21$

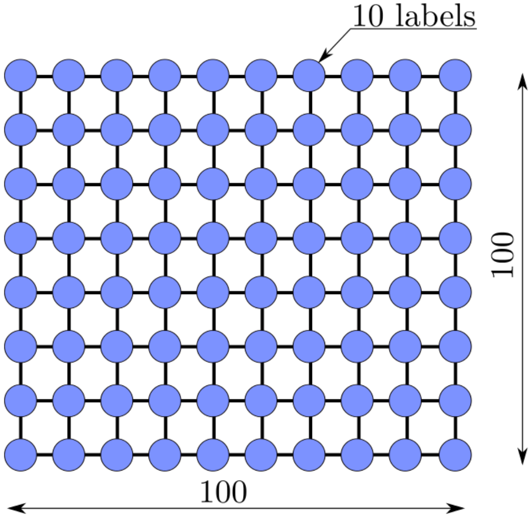
?

Dual MAP LP Objective: Number of Variables

Dual: $$\max_{\phi \in \mathbb{R}^J} \sum_{v \in \mathcal{V}} \min_{s \in \mathcal{X}_v} \theta_v^{\phi}(s) + \sum_{uv \in \mathcal{E}} \min_{(s,t) \in \mathcal{X}_{uv}} \theta_{uv}^{\phi}(s,t)$$



1) 10^5 and less
2) 10^6
3) 10^7
4) 10^8
5) 10^9
6) 10^{10} and more

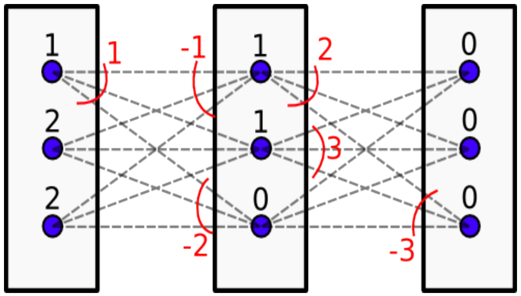


Pascal VOC

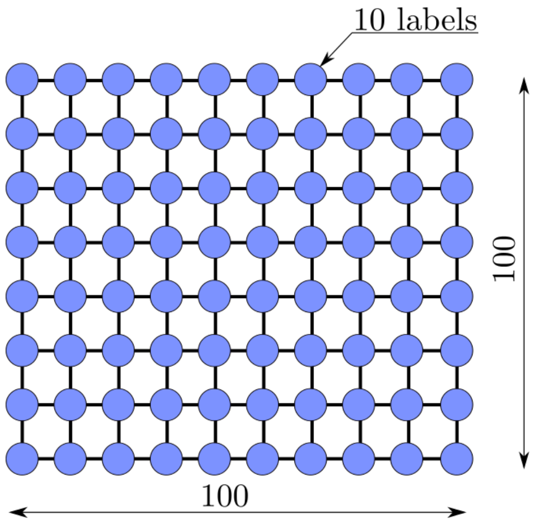
$500 \times 300 \times 21$

?

Dual: $$\max_{\phi\in\mathbb{R}^J}\sum_{v\in\mathcal{V}}\min_{s\in\mathcal{X}_v}\theta_v^\phi(s)+\sum_{uv\in\mathcal{E}}\min_{(s,t)\in\mathcal{X}_{uv}}\theta_{uv}^\phi(s,t)$$



Dual is preferable to optimize.



primal 10^6, dual 10^5

Pascal VOC

$500\times300\times21$

Primal 10^9, dual 10^7