D5: Databases and Information Systems
Data Mining and Matrices, SS 2015
Homework #1: Prerequisites
Due: **5 May 2015**

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You can discuss these problems with other students, but every body must hand in their own answers. You can use computers etc. to perform the algebraic operations, but you must show the intermediate steps (and "computer said so" is never a valid answer). Remember to write your name and matriculation number to every answer sheet!

Problem 1 (Linear independency). Consider the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 5 & 0 & 0 & 7 \\ 1 & 3 & 1 & 0 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 2 & 2 \\ 1 & 2 & 7 & 5 & 7 & 0 & 0 \\ 1 & 5 & 3 & 1 & 2 & 2 & 7 \\ 0 & 3 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Compute the number of linearly independent rows of $\boldsymbol{A}$ by reducing it to the reduced row echelon form (using the Gauss–Jordan elimination). Your answer should show at least five steps of the elimination process, the final result, and clearly state the number of linearly independent rows.

Problem 2 (Matrix inverse). Consider the following matrices. Do they have inverses. Briefly motivate your answer (i.e. just "yes" or "no" is not enough for an answer).

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 5 & 0 & 0 & 7 \\ 1 & 3 & 1 & 0 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 2 & 2 \\ 1 & 2 & 7 & 5 & 7 & 0 & 0 \\ 1 & 5 & 3 & 1 & 2 & 2 & 7 \\ 0 & 3 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 2 & 0 & 5 & 0 & 0 & 7 \\ 1 & 3 & 1 & 0 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 2 & 2 \\ 1 & 5 & 3 & 1 & 2 & 2 & 7 \\ 0 & 3 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 2 & 0 & 5 & 0 & 0 & 7 \\ 1 & 2 & 0 & 5 & 0 & 0 & 7 \\ 0 & 3 & 1 & 0 & 0 & 2 & 2 \\ 1 & 2 & 7 & 5 & 7 & 0 & 0 \\ 1 & 5 & 3 & 1 & 2 & 2 & 7 \\ 0 & 3 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 5 & 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Problem 3 (System of linear equations). Solve the following system of linear equations. Your answer should contain sufficient details explaining how you reached the solution; just giving the solution is not enough.

$$1x_1 + 2x_2 + 0x_3 + 5x_4 + 0x_5 + 0x_6 + 7x_7 = 97\tfrac{1}{3} \tag{3.1}$$
$$1x_1 + 3x_2 + 1x_3 + 0x_4 + 2x_5 + 0x_6 + 0x_7 = 42\tfrac{2}{3} \tag{3.2}$$
$$0x_1 + 3x_2 + 1x_3 + 0x_4 + 0x_5 + 2x_6 + 2x_7 = 29 \tag{3.3}$$
$$1x_1 + 2x_2 + 7x_3 + 5x_4 + 7x_5 + 0x_6 + 0x_7 = 190\tfrac{2}{3} \tag{3.4}$$
$$1x_1 + 5x_2 + 3x_3 + 1x_4 + 2x_5 + 2x_6 + 7x_7 = 106\tfrac{1}{3} \tag{3.5}$$
$$0x_1 + 3x_2 + 0x_3 + 5x_4 + 0x_5 + 0x_6 + 0x_7 = 70\tfrac{1}{3} \tag{3.6}$$
$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 1x_7 = 4 \tag{3.7}$$

Problem 4 (Norms). A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *norm* if it satisfies the following properties for all $a \in \mathbb{R}$ and all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$f(a\boldsymbol{x}) = |a|\, f(\boldsymbol{x}) \tag{4.1}$$
$$f(\boldsymbol{x} + \boldsymbol{y}) \le f(\boldsymbol{x}) + f(\boldsymbol{y}) \tag{4.2}$$
$$\text{If } f(\boldsymbol{x}) = 0 \text{ then } \boldsymbol{x} \text{ is all-zeros vector.} \tag{4.3}$$

Proof that if f is a norm, then

a) if $\mathbf{0} \in \mathbb{R}^n$ is the all-zeros vector, then $f(\mathbf{0}) = 0$

b) f is always non-negative (that is, $f(\boldsymbol{x}) \ge 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$).

Problem 5 (The L_1 norm). Let $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}$ be the L_1 vector norm, that is,

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i| \tag{5.1}$$

for all $\boldsymbol{x} = (x_i)_{i=1}^n \in \mathbb{R}^n$. Proof that $\|\cdot\|_1$ is a norm (i.e. it satisfies properties (4.1)–(4.3)).

Problem 6 (The Frobenius norm). The *trace* function tr computes the sum of the diagonal elements of a square matrix, that is, if $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, then

$$\operatorname{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii} \ . \tag{6.1}$$

The *Frobenius norm* of an n-by-m real matrix $\boldsymbol{B}$ is defined as

$$\|\boldsymbol{B}\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij}^2 \right)^{1/2} \ . \tag{6.2}$$

Proof that

$$\|\boldsymbol{B}\|_F = \left(\operatorname{tr}(\boldsymbol{B}^T \boldsymbol{B})\right)^{1/2} , \tag{6.3}$$

where $\boldsymbol{B}^T$ is the transpose of $\boldsymbol{B}$.