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You can discuss these problems with other students, but everybody must hand in their own answers. You can use computers etc. to perform the algebraic operations, but you must show the intermediate steps (and "computer said so" is never a valid answer). You can return either computer-typeset solutions by email (but no scanned or photographed solutions are accepted), or legibly hand-written or computer-typeset solutions personally to the lecture. Notice that the DL is strict. *Remember to write your name and matriculation number to every answer sheet! If you want to discuss the solutions with the tutor, the tutorial meeting is the time to do that. If you cannot attend the tutorial meeting but want to discuss with the tutor, you must schedule a meeting with the tutor via email.*

Problem 1 (Traces and eigenvalues). Let $\boldsymbol{A} = \boldsymbol{Q\Lambda Q}^T \in \mathbb{R}^{n\times n}$ be a matrix and its eigen-decomposition. You can assume that $\boldsymbol{Q}$ is orthogonal and that the eigenvalues are real. Show that

$$\mathrm{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i = \mathrm{tr}(\boldsymbol{\Lambda}) \ . \tag{1.1}$$

Problem 2 (Laplacian is positive semi-definite). Let $G = (V, E)$ be an undirected graph and let $\boldsymbol{A}$ be its adjacency matrix and $\boldsymbol{\Delta}$ its degree matrix. Let $\boldsymbol{L} = \boldsymbol{\Delta} - \boldsymbol{A}$ be the Laplacian of G. Recall that the *incidence matrix* $\boldsymbol{P}$ of G (for some fixed but arbitrary ordering of the edges) is the $|V|$-by-$|E|$ matrix with

$$p_{ij} = \begin{cases} 1 & \text{if edge } j \text{ starts from node } i \\ -1 & \text{if edge } j \text{ ends to node } i \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Show that

$$\boldsymbol{L} = \boldsymbol{P}\boldsymbol{P}^T \tag{2.2}$$

and conclude that the Laplacian is positive semi-definite.

Problem 3 (Normalized cut). Show that the solution for the relaxed normalized cut is obtained by taking the k least eigenvectors of the symmetric normalized Laplacian L^s similarly as the ratio cut is solved by taking the k least eigenvectors of the Laplacian.

Hint: Express normalized cut using the symmetric Laplacian by re-writing the equation

$$J_{nc}(\mathcal{C}) = \sum_{i=1}^{k} \frac{\boldsymbol{c}_i^T \boldsymbol{L} \boldsymbol{c}_i}{\boldsymbol{c}_i^T \boldsymbol{\Delta} \boldsymbol{c}_i}$$

using the facts that $\boldsymbol{\Delta} = \boldsymbol{\Delta}^{1/2}\boldsymbol{\Delta}^{1/2}$, $\boldsymbol{\Delta}^{1/2}\boldsymbol{\Delta}^{-1/2} = I$, and $\boldsymbol{\Delta} = \boldsymbol{\Delta}^T$ (as $\boldsymbol{\Delta}$ is diagonal).

Problem 4 (Boolean orthogonal matrices). We can define the *Boolean length* of a vector $\boldsymbol{v} \in \{0,1\}^n$ as $\|\boldsymbol{v}\|_B = \bigvee_{i=1}^n v_i$. Similarly, we can define the *Boolean inner product* of two Boolean vectors $\boldsymbol{v}, \boldsymbol{u} \in \{0,1\}^n$ as $\langle \boldsymbol{v}, \boldsymbol{u}\rangle_B = \bigvee_{i=1}^n v_i \wedge u_i$. We say two Boolean vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are *Boolean orthogonal* if $\langle \boldsymbol{v}, \boldsymbol{u}\rangle_B = 0$, and we say that a Boolean matrix $\boldsymbol{M} \in \{0,1\}^{n\times n}$ is *Boolean orthogonal* if all its columns are mutually orthogonal (i.e. $\langle \boldsymbol{m}_i, \boldsymbol{m}_j\rangle_B = 0$ for $i \neq j$) and have unit Boolean length (i.e. $\|\boldsymbol{m}_i\|_B = 1$ for all i).

A *permutation matrix* $\mathbf{\Pi} \in \{0,1\}^{n \times n}$ is a square binary matrix where every column and every row has exactly one 1. Show that permutation matrices are Boolean orthogonal and that they are the only Boolean orthogonal matrices.

Problem 5 (Size of matrix with bounded Boolean rank). Let $\boldsymbol{A} \in \{0,1\}^{n \times n}$ be a Boolean matrix with distinct rows and columns. Show that

$$\operatorname{rank}_B(\boldsymbol{A}) \geq \log_2 n \; . \tag{5.1}$$

The next part is optional: you can earn an extra point if you answer to it, also.

Assume we are given some target rank k and Boolean matrix $\boldsymbol{A} \in \{0,1\}^{n \times m}$ and we need to answer to the question "Is the Boolean rank of $\boldsymbol{A}$ exactly k?" Show that we can answer to this question in time which depends only on k and not on the original size of the matrix (i.e. n or m).

Hint: Bound the maximum size of the Boolean matrix that can have Boolean rank k using a function that depends only on k. If the matrix is larger than this, the answer is always no; if the matrix is smaller than this, show how you can find the exact decomposition in time that depends only on k.

Problem 6 (Combining approximate rank-1 components). Let $\boldsymbol{A} \in \{0,1\}^{n \times m}$ be a Boolean matrix. One approach to find an approximate Boolean rank-k decomposition of $\boldsymbol{A}$ is to first find a collection of binary rank-1 matrices and then select k of those. Let $C = \{\boldsymbol{b}_1\boldsymbol{c}_1^T, \boldsymbol{b}_2\boldsymbol{c}_2^T, \ldots, \boldsymbol{b}_r\boldsymbol{c}_r^T\}$ be a set of r rank-1 Binary n-by-m matrices (i.e. $\boldsymbol{b}_i \in \{0,1\}^n$ and $\boldsymbol{c}_i \in \{0,1\}^m$). Now our task is to find k ($k < r$) rank-1 matrices from C to set S such that

$$\left\| \boldsymbol{A} - \left(\bigvee_{\boldsymbol{b}\boldsymbol{c}^T \in S} \boldsymbol{b}\boldsymbol{c}^T \right) \right\|_F^2 \tag{6.1}$$

is minimized.

Assume we have an algorithm to solve the Positive–Negative Partial Set Cover problem (±PSC). Show how we can use that algorithm to solve the above problem.