Problem 1 (Khatri–Rao is associative). Let $A \in \mathbb{R}^{I \times L}$, $B \in \mathbb{R}^{J \times L}$, and $C \in \mathbb{R}^{K \times L}$. Show that

$$(A \odot B) \odot C = A \odot (B \odot C).$$

Solution. Let $D = (A \odot B)$ so that $d_{ij \ell} = a_{i \ell} b_{j \ell}$. Then

$$D \odot C = [d_1 \odot c_1, d_2 \odot c_2, \ldots, d_L \odot c_K]$$

and

$$(D \odot C)_{(i,j,k)\ell} = d_{ij \ell} c_{k \ell} = a_{i \ell} b_{j \ell} c_{k \ell} = a_{i \ell} (b_{j \ell} \otimes c_k)_{(j,k)\ell} = (A \odot (B \odot C))_{(i,j,k)\ell},$$

proving the claim.
**Problem 2** (Khatri–Rao and pseudo-inverse). Recall that the Moore–Penrose pseudo-inverse of a matrix $M$ is a matrix $M^+$ such that

\[
MM^+M = M \quad \text{(2.1)}
\]
\[
M^+MM^+ = M^+ \quad \text{(2.2)}
\]
\[
(MM^+)^T = MM^+ \quad \text{(2.3)}
\]
\[
(M^+M)^T = M^+M \quad \text{(2.4)}
\]

Let $A \in \mathbb{R}^{I \times K}$ and $B \in \mathbb{R}^{J \times K}$ be such that $K < \min\{I, J\}$ and rank$((A^T A) \ast (B^T B)) = K$. Show that

\[
(A \odot B)^+ = ((A^T A) \ast (B^T B))^+ (A \odot B)^T. \quad \text{(2.5)}
\]

**Hint:** You can use the equation

\[
(A \odot B)^T (A \odot B) = A^T A \ast B^T B. \quad \text{(2.6)}
\]

**Solution.** To prove the claim, we need to show that setting $M = A \odot B$ and $M^+ = ((A^T A) \ast (B^T B))^+$ satisfies (2.1)–(2.2). We start from (2.1):

\[
(A \odot B)(A \odot B)^+ (A \odot B) = (A \odot B)((A^T A) \ast (B^T B))^+(A \odot B)^T (A \odot B)
\]
\[
= (A \odot B)((A^T A) \ast (B^T B))^+(A \odot B) = (A \odot B),
\]

where the least equality is due to the fact that $(A^T A) \ast (B^T B)$ is invertible and hence $((A^T A) \ast (B^T B))^+ = ((A^T A) \ast (B^T B))^{-1}$.

Equation (2.2) follows similarly. For (2.3), we write

\[
((A \odot B)(A \odot B)^+)^T = ((A \odot B)((A^T A) \ast (B^T B))^+(A \odot B)^T)^T
\]
\[
= ((A^T A) \ast (B^T B))^+(A \odot B)^T (A \odot B)^T
\]
\[
= (A \odot B)((A^T A) \ast (B^T B))^+(A \odot B)^T
\]
\[
= (A \odot B)((A^T A) \ast (B^T B))^+(A \odot B)^T
\]
\[
= (A \odot B)(A \odot B)^+,
\]

where the penultimate equation follows from the fact that $(A^T A) \ast (B^T B)$ is a symmetric matrix. Equation (2.4) follows more easily from the properties of the pseudo-inverse:

\[
((A \odot B)^+(A \odot B))^T = (((A^T A) \ast (B^T B))^+(A \odot B)^T(A \odot B))^T
\]
\[
= (((A \odot B)^T(A \odot B))^T(A \odot B))^T
\]
\[
= ((A \odot B)^T(A \odot B))^+(A \odot B)^T(A \odot B)
\]
\[
= (A \odot B)^+(A \odot B).
\]
Problem 3 (CP decomposition). Let

\[ A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}. \]

a) Calculate \( a_1 \odot b_1 \odot c_1 \).
b) Calculate \( a_2 \odot b_2 \odot c_2 \).
c) Calculate \( \llbracket A, B, C \rrbracket \).

Solution.

a) Calling the result \( \mathbf{T}^{(1)} \), the frontal slices are

\[ T_1^{(1)} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \quad \text{and} \quad T_2^{(1)} = \begin{pmatrix} -2 & -4 & -6 \\ -4 & -8 & -12 \\ -6 & -12 & -18 \end{pmatrix}. \]

b) Calling the result \( \mathbf{T}^{(2)} \), the frontal slices are

\[ T_1^{(2)} = \begin{pmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{pmatrix} \quad \text{and} \quad T_2^{(2)} = \begin{pmatrix} -8 & -16 & -24 \\ -10 & -20 & -30 \\ -12 & -24 & -36 \end{pmatrix}. \]

c) The result is

\[ \llbracket A, B, C \rrbracket = a_1 \odot b_1 \odot c_1 + a_2 \odot b_2 \odot c_2 \]

with

\[ [A, B, C]_1 = \begin{pmatrix} 5 & 10 & 15 \\ 7 & 14 & 21 \\ 9 & 18 & 27 \end{pmatrix}, \quad [A, B, C]_2 = \begin{pmatrix} -10 & -20 & -30 \\ -14 & -28 & -42 \\ -18 & -36 & -54 \end{pmatrix}. \]
**Problem 4** (Uniqueness of a rank decomposition). In the lectures we saw a tensor $\mathbf{T} \in \mathbb{R}^{2 \times 2 \times 2}$, 

$$
\mathbf{T}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
$$

that has real tensor rank 3. One factorization that obtains this rank 3 is 

$$
\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.
$$

What can you say about the uniqueness of this factorization?

**Solution.** First, $\text{rank}(\mathbf{A} \odot \mathbf{B}) = \text{rank}(\mathbf{A} \odot \mathbf{C}) = \text{rank}(\mathbf{B} \odot \mathbf{C}) = 3$, so the first necessary condition is fulfilled.

On the other hand, it is easy to see that the $k$-rank of the factor matrices is 2 for all of them. Hence 

$$
k_{\mathbf{A}} + k_{\mathbf{A}} + k_{\mathbf{A}} = 6 < 2 \cdot 3 + 2 = 8,
$$

and the sufficient condition of the uniqueness isn’t fulfilled. Reading Kolda & Bader, we learn that this sufficient condition is also necessary for tensors with rank 2 or 3. Hence the factorization is not unique.