D5: Databases and Information Systems
Tensors in Data Analysis, WS 2017–18
Homework #4: Tucker decompositions
Tutorial: **13 October 2017** at 14:30

You can discuss these problems with other students, but everybody must do and present their own answers. You can use computers etc. to perform the algebraic operations, but you must show the intermediate steps (and "computer said so" is never a valid answer). You are of course free to use material from the Internet, but again, you must present the intermediate steps and you must also be able to explain why the steps are valid and why you chose them. You can mark an answer even if it is not complete or correct, as long as you have made significant progress towards solving it. Note, however, that the lecturer does the final decision on whether your solution is complete (or correct) enough for a mark.

Problem 1 (Tucker1). In the lecture it was stated that the Tucker1 decomposition $[\![\boldsymbol{\mathcal{G}}; \boldsymbol{A}, \boldsymbol{I}, \boldsymbol{I}]\!]$ such that $\|\{\| \boldsymbol{\mathcal{T}} - [\![\boldsymbol{\mathcal{G}}; \boldsymbol{A}, \boldsymbol{I}, \boldsymbol{I}]\!]\}$ is equivalent to standard least-squares matrix factorization. Show that this is the case.

Problem 2 (Tucker3). Let $\boldsymbol{\mathcal{G}}$ be a 2-by-2-by-2 defined by its frontal slices as

$$\boldsymbol{G}_1 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{G}_2 = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} ,$$

and let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} -2 & 2 \\ 3 & -3 \\ -4 & -4 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Calculate $\boldsymbol{\mathcal{G}} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$.

Problem 3 (Inverses in tensor-matrix product). Let $\boldsymbol{\mathcal{G}} \in \mathbb{R}^{P \times Q \times R}$ with $P \leqslant I$, $Q \leqslant J$, and $R \leqslant K$, and let $\boldsymbol{A} \in \mathbb{R}^{I \times P}$, $\boldsymbol{B} \in \mathbb{R}^{J \times Q}$, and $\boldsymbol{C} \in \mathbb{R}^{K \times R}$. Assume that $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are column-orthogonal, that is, $\boldsymbol{A}^T \boldsymbol{A} = \boldsymbol{I}$ etc.

Let $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{G}} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$. Prove that

$$\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{T}} \times_1 \boldsymbol{A}^T \times_2 \boldsymbol{B}^T \times_3 \boldsymbol{C}^T .$$

Problem 4 (Vectorization and Kronecker). To solve

$$\operatorname*{arg\,min}_{\boldsymbol{G}_k} \left\| \boldsymbol{T}_k - \boldsymbol{A} \boldsymbol{G}_k \boldsymbol{A}^T \right\| ,$$

we wrote it as

$$\operatorname*{arg\,min}_{\boldsymbol{G}_k} \left\| \operatorname{vec}(\boldsymbol{T}_{(k)}) - (\boldsymbol{A} \otimes \boldsymbol{A}) \operatorname{vec}(\boldsymbol{G}_k) \right\| .$$

Prove that this re-writing is correct. That is, show that for any matrices $\boldsymbol{A} \in \mathbb{R}^{I \times K}$ and $\boldsymbol{B} \in \mathbb{R}^{K \times K}$, we have

$$\operatorname{vec}(\boldsymbol{A} \boldsymbol{B} \boldsymbol{A}^T) = (\boldsymbol{A} \otimes \boldsymbol{A}) \operatorname{vec}(\boldsymbol{B}) .$$