Problem 1 (Tucker1). In the lecture it was stated that the Tucker1 decomposition \([\mathcal{G}; A, I, I]\) such that \(\|T - [\mathcal{G}; A, I, I]\|\) is equivalent to standard least-squares matrix factorization. Show that this is the case.

Solution. The mode-1 matricization is

\[\|T_{(1)} - AG_{(1)}(I \otimes I)^T\| = \|T_{(1)} - AG_{(1)}\|,\]

as \(I_J \otimes I_K = I_{JK}\). Hence, we can solve the problem perfectly by looking only at the mode-1 matricization.
Problem 2 (Tucker3). Let $G$ be a 2-by-2-by-2 defined by its frontal slices as
\[
G_1 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad G_2 = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix},
\]
and let
\[
A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 2 \\ 3 & -3 \\ -4 & -4 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
Calculate $G \times_1 A \times_2 B \times_3 C$.

Solution. Let $T = G \times_1 A \times_2 B \times_3 C$. The frontal slices of $T$ are
\[
T_1 = \begin{pmatrix} 26 & -39 & 4 \\ 34 & -51 & -4 \\ 42 & -63 & -12 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 14 & -21 & -44 \\ 22 & -33 & -52 \\ 30 & -45 & -60 \end{pmatrix}.
\]
Problem 3 (Inverses in tensor-matrix product). Let \( G \in \mathbb{R}^{P \times Q \times R} \) with \( P \leq I \), \( Q \leq J \), and \( R \leq K \), and let \( A \in \mathbb{R}^{I \times P} \), \( B \in \mathbb{R}^{J \times Q} \), and \( C \in \mathbb{R}^{K \times R} \). Assume that \( A \), \( B \), and \( C \) are column-orthogonal, that is, \( A^T A = I \) etc.

Let \( T = G \times_1 A \times_2 B \times_3 C \). Prove that

\[
G = T \times_1 A^T \times_2 B^T \times_3 C^T.
\]

Solution.

\[
G = G \times_1 (A^T A) \times_2 (B^T B) \times_3 (C^T C) \\
= (G \times_1 A \times_2 B \times_3 C) \times_1 A^T \times_2 B^T \times_3 C^T \\
= T \times_1 A^T \times_2 B^T \times_3 C^T.
\]
**Problem 4** (Vectorization and Kronecker). To solve

\[
\arg\min_{G_k} \| T_k - AG_kA^T \| .
\]

we wrote it as

\[
\arg\min_{G_k} \| \text{vec}(T_k) - (A \otimes A) \text{vec}(G_k) \| .
\]

Prove that this re-writing is correct. That is, show that for any matrices \( A \in \mathbb{R}^{I \times K} \) and \( B \in \mathbb{R}^{K \times K} \), we have

\[\text{vec}(ABA^T) = (A \otimes A) \text{vec}(B).\]

**Solution.**

\[
\begin{align*}
\text{vec}(ABA^T) &= \\
&= \begin{pmatrix}
(ABA^T)(:,1) \\
(ABA^T)(:,2) \\
\vdots \\
(ABA^T)(:,I)
\end{pmatrix} \\
&= \begin{pmatrix}
\langle A(1,:), (BA^T)(:,1) \rangle \\
\langle A(2,:), (BA^T)(:,1) \rangle \\
\vdots \\
\langle A(I,:), (BA^T)(:,1) \rangle \\
\langle A(1,:), (BA^T)(:,2) \rangle \\
\vdots \\
\langle A(I,:), (BA^T)(:,2) \rangle \\
\vdots \\
\langle A(1,:), (BA^T)(:,I) \rangle \\
\vdots \\
\langle A(I,:), (BA^T)(:,I) \rangle
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&= a_{11}\langle A(1,:), B(1,:) \rangle + a_{12}\langle A(1,:), B(2,:) \rangle + \cdots + a_{1K}\langle A(1,:), B(K,:) \rangle \\
&+ a_{21}\langle A(1,:), B(1,:) \rangle + a_{22}\langle A(1,:), B(2,:) \rangle + \cdots + a_{2K}\langle A(1,:), B(K,:) \rangle \\
&+ \cdots \\
&+ a_{I1}\langle A(I,:), B(1,:) \rangle + a_{I2}\langle A(I,:), B(2,:) \rangle + \cdots + a_{IK}\langle A(I,:), B(K,:) \rangle \\
&= \begin{pmatrix}
a_{11}A & a_{12}A & \cdots & B(:,1) \\
a_{21}A & a_{22}A & \cdots & B(:,2) \\
\vdots & \vdots & \ddots & \vdots \\
\end{pmatrix} = (A \otimes A) \text{vec}(B)
\end{align*}
\]