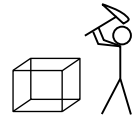


**Problem 1** (Tucker1). In the lecture it was stated that the Tucker1 decomposition  $[[\mathcal{G}; \mathbf{A}, \mathbf{I}, \mathbf{I}]]$  such that  $\|\mathcal{T} - [[\mathcal{G}; \mathbf{A}, \mathbf{I}, \mathbf{I}]]\|$  is equivalent to standard least-squares matrix factorization. Show that this is the case.

*Solution.* The mode-1 matricization is

$$\|\mathbf{T}_{(1)} - \mathbf{A}\mathbf{G}_{(1)}(\mathbf{I} \otimes \mathbf{I})^T\| = \|\mathbf{T}_{(1)} - \mathbf{A}\mathbf{G}_{(1)}\| ,$$

as  $\mathbf{I}_J \otimes \mathbf{I}_K = \mathbf{I}_{JK}$ . Hence, we can solve the problem perfectly by looking only at the mode-1 matricization.



**Problem 2** (Tucker3). Let  $\mathcal{G}$  be a 2-by-2-by-2 defined by its frontal slices as

$$\mathbf{G}_1 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{G}_2 = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix},$$

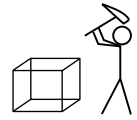
and let

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -2 & 2 \\ 3 & -3 \\ -4 & -4 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Calculate  $\mathcal{G} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C}$ .

*Solution.* Let  $\mathcal{T} = \mathcal{G} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C}$ . The frontal slices of  $\mathcal{T}$  are

$$\mathbf{T}_1 = \begin{pmatrix} 26 & -39 & 4 \\ 34 & -51 & -4 \\ 42 & -63 & -12 \end{pmatrix} \quad \text{and} \quad \mathbf{T}_2 = \begin{pmatrix} 14 & -21 & -44 \\ 22 & -33 & -52 \\ 30 & -45 & -60 \end{pmatrix}.$$



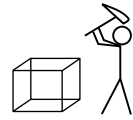
**Problem 3** (Inverses in tensor-matrix product). Let  $\mathcal{G} \in \mathbb{R}^{P \times Q \times R}$  with  $P \leq I$ ,  $Q \leq J$ , and  $R \leq K$ , and let  $\mathbf{A} \in \mathbb{R}^{I \times P}$ ,  $\mathbf{B} \in \mathbb{R}^{J \times Q}$ , and  $\mathbf{C} \in \mathbb{R}^{K \times R}$ . Assume that  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are column-orthogonal, that is,  $\mathbf{A}^T \mathbf{A} = \mathbf{I}$  etc.

Let  $\mathcal{T} = \mathcal{G} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C}$ . Prove that

$$\mathcal{G} = \mathcal{T} \times_1 \mathbf{A}^T \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T .$$

*Solution.*

$$\begin{aligned} \mathcal{G} &= \mathcal{G} \times_1 (\mathbf{A}^T \mathbf{A}) \times_2 (\mathbf{B}^T \mathbf{B}) \times_3 (\mathbf{C}^T \mathbf{C}) \\ &= (\mathcal{G} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C}) \times_1 \mathbf{A}^T \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T \\ &= \mathcal{T} \times_1 \mathbf{A}^T \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T . \end{aligned}$$



**Problem 4** (Vectorization and Kronecker). To solve

$$\arg \min_{\mathbf{G}_k} \left\| \mathbf{T}_k - \mathbf{A} \mathbf{G}_k \mathbf{A}^T \right\| ,$$

we wrote it as

$$\arg \min_{\mathbf{G}_k} \left\| \text{vec}(\mathbf{T}_{(k)}) - (\mathbf{A} \otimes \mathbf{A}) \text{vec}(\mathbf{G}_k) \right\| .$$

Prove that this re-writing is correct. That is, show that for any matrices  $\mathbf{A} \in \mathbb{R}^{I \times K}$  and  $\mathbf{B} \in \mathbb{R}^{K \times K}$ , we have

$$\text{vec}(\mathbf{A} \mathbf{B} \mathbf{A}^T) = (\mathbf{A} \otimes \mathbf{A}) \text{vec}(\mathbf{B}) .$$

*Solution.*

$$\begin{aligned} \text{vec}(\mathbf{A} \mathbf{B} \mathbf{A}^T) &= \begin{pmatrix} (\mathbf{A} \mathbf{B} \mathbf{A}^T)(:, 1) \\ (\mathbf{A} \mathbf{B} \mathbf{A}^T)(:, 2) \\ \vdots \\ (\mathbf{A} \mathbf{B} \mathbf{A}^T)(:, I) \end{pmatrix} = \begin{pmatrix} \langle \mathbf{A}(1, :), (\mathbf{B} \mathbf{A}^T)(:, 1) \rangle \\ \langle \mathbf{A}(2, :), (\mathbf{B} \mathbf{A}^T)(:, 1) \rangle \\ \vdots \\ \langle \mathbf{A}(I, :), (\mathbf{B} \mathbf{A}^T)(:, 1) \rangle \\ \langle \mathbf{A}(1, :), (\mathbf{B} \mathbf{A}^T)(:, 2) \rangle \\ \vdots \\ \langle \mathbf{A}(I, :), (\mathbf{B} \mathbf{A}^T)(:, I) \rangle \end{pmatrix} \\ &= \begin{pmatrix} a_{11} \langle \mathbf{A}(1, :), \mathbf{B}(1, :) \rangle + a_{12} \langle \mathbf{A}(1, :), \mathbf{B}(2, :) \rangle + \cdots + a_{1K} \langle \mathbf{A}(1, :), \mathbf{B}(K, :) \rangle \\ a_{21} \langle \mathbf{A}(1, :), \mathbf{B}(1, :) \rangle + a_{22} \langle \mathbf{A}(1, :), \mathbf{B}(2, :) \rangle + \cdots + a_{2K} \langle \mathbf{A}(1, :), \mathbf{B}(K, :) \rangle \\ \vdots \\ a_{I1} \langle \mathbf{A}(1, :), \mathbf{B}(1, :) \rangle + a_{I2} \langle \mathbf{A}(1, :), \mathbf{B}(2, :) \rangle + \cdots + a_{IK} \langle \mathbf{A}(1, :), \mathbf{B}(K, :) \rangle \\ a_{11} \langle \mathbf{A}(2, :), \mathbf{B}(1, :) \rangle + a_{12} \langle \mathbf{A}(2, :), \mathbf{B}(2, :) \rangle + \cdots + a_{1K} \langle \mathbf{A}(2, :), \mathbf{B}(K, :) \rangle \\ \vdots \\ a_{I1} \langle \mathbf{A}(I, :), \mathbf{B}(1, :) \rangle + a_{I2} \langle \mathbf{A}(I, :), \mathbf{B}(2, :) \rangle + \cdots + a_{IK} \langle \mathbf{A}(I, :), \mathbf{B}(K, :) \rangle \end{pmatrix} \\ &= \begin{pmatrix} a_{11} \mathbf{A} & a_{12} \mathbf{A} & \cdots \\ a_{21} \mathbf{A} & a_{22} \mathbf{A} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \mathbf{B}(:, 1) \\ \mathbf{B}(:, 2) \\ \vdots \\ \mathbf{B}(:, K) \end{pmatrix} = (\mathbf{A} \otimes \mathbf{A}) \text{vec}(\mathbf{B}) \end{aligned}$$