D5: Databases and Information Systems
Tensors in Data Analysis, WS 2017–18
Homework #4: Tucker decompositions
Tutorial: **13 October 2017** at 14:30

**Problem 1** (Tucker1). In the lecture it was stated that the Tucker1 decomposition $[\![\boldsymbol{\mathcal{G}}; \boldsymbol{A}, \boldsymbol{I}, \boldsymbol{I}]\!]$ such that $\|\boldsymbol{\mathcal{T}} - [\![\boldsymbol{\mathcal{G}}; \boldsymbol{A}, \boldsymbol{I}, \boldsymbol{I}]\!]\|$ is equivalent to standard least-squares matrix factorization. Show that this is the case.

*Solution.* The mode-1 matricization is

$$\left\|\boldsymbol{T}_{(1)} - \boldsymbol{A}\boldsymbol{G}_{(1)}(\boldsymbol{I} \otimes \boldsymbol{I})^T\right\| = \left\|\boldsymbol{T}_{(1)} - \boldsymbol{A}\boldsymbol{G}_{(1)}\right\| \ ,$$

as $\boldsymbol{I}_J \otimes \boldsymbol{I}_K = \boldsymbol{I}_{JK}$. Hence, we can solve the problem perfectly by looking only at the mode-1 matricization.

**Problem 2** (Tucker3). Let $\boldsymbol{\mathcal{G}}$ be a 2-by-2-by-2 defined by its frontal slices as

$$\boldsymbol{G}_1 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{G}_2 = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} ,$$

and let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} -2 & 2 \\ 3 & -3 \\ -4 & -4 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Calculate $\boldsymbol{\mathcal{G}} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$.

*Solution.* Let $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{G}} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$. The frontal slices of $\boldsymbol{\mathcal{T}}$ are

$$\boldsymbol{T}_1 = \begin{pmatrix} 26 & -39 & 4 \\ 34 & -51 & -4 \\ 42 & -63 & -12 \end{pmatrix} \quad \text{and} \quad \boldsymbol{T}_2 = \begin{pmatrix} 14 & -21 & -44 \\ 22 & -33 & -52 \\ 30 & -45 & -60 \end{pmatrix} .$$

**Problem 3** (Inverses in tensor-matrix product). Let $\boldsymbol{\mathcal{G}} \in \mathbb{R}^{P \times Q \times R}$ with $P \leqslant I$, $Q \leqslant J$, and $R \leqslant K$, and let $\boldsymbol{A} \in \mathbb{R}^{I \times P}$, $\boldsymbol{B} \in \mathbb{R}^{J \times Q}$, and $\boldsymbol{C} \in \mathbb{R}^{K \times R}$. Assume that $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are column-orthogonal, that is, $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$ etc.

Let $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{G}} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$. Prove that

$$\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{T}} \times_1 \boldsymbol{A}^T \times_2 \boldsymbol{B}^T \times_3 \boldsymbol{C}^T \ .$$

*Solution.*

$$\begin{aligned}
\boldsymbol{\mathcal{G}} &= \boldsymbol{\mathcal{G}} \times_1 (\boldsymbol{A}^T\boldsymbol{A}) \times_2 (\boldsymbol{B}^T\boldsymbol{B}) \times_3 (\boldsymbol{C}^T\boldsymbol{C}) \\
&= (\boldsymbol{\mathcal{G}} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}) \times_1 \boldsymbol{A}^T \times_2 \boldsymbol{B}^T \times_3 \boldsymbol{C}^T \\
&= \boldsymbol{\mathcal{T}} \times_1 \boldsymbol{A}^T \times_2 \boldsymbol{B}^T \times_3 \boldsymbol{C}^T \ .
\end{aligned}$$

**Problem 4** (Vectorization and Kronecker). To solve

$$\arg\min_{\boldsymbol{G}_k} \left\| \boldsymbol{T}_k - \boldsymbol{A}\boldsymbol{G}_k\boldsymbol{A}^T \right\| ,$$

we wrote it as

$$\arg\min_{\boldsymbol{G}_k} \left\| \mathrm{vec}(\boldsymbol{T}_{(k)}) - (\boldsymbol{A} \otimes \boldsymbol{A})\,\mathrm{vec}(\boldsymbol{G}_k) \right\| .$$

Prove that this re-writing is correct. That is, show that for any matrices $\boldsymbol{A} \in \mathbb{R}^{I \times K}$ and $\boldsymbol{B} \in \mathbb{R}^{K \times K}$, we have

$$\mathrm{vec}(\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^T) = (\boldsymbol{A} \otimes \boldsymbol{A})\,\mathrm{vec}(\boldsymbol{B}) .$$

*Solution.*

$$\mathrm{vec}(\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^T) = \begin{pmatrix} (\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^T)(:,1) \\ (\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^T)(;,2) \\ \vdots \\ (\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^T)(:,I) \end{pmatrix} = \begin{pmatrix} \langle \boldsymbol{A}(1,:), (\boldsymbol{B}\boldsymbol{A}^T)(:,1) \rangle \\ \langle \boldsymbol{A}(2,:), (\boldsymbol{B}\boldsymbol{A}^T)(:,1) \rangle \\ \vdots \\ \langle \boldsymbol{A}(I,:), (\boldsymbol{B}\boldsymbol{A}^T)(:,1) \rangle \\ \langle \boldsymbol{A}(1,:), (\boldsymbol{B}\boldsymbol{A}^T)(:,2) \rangle \\ \vdots \\ \langle \boldsymbol{A}(I,:), (\boldsymbol{B}\boldsymbol{A}^T)(:,I) \rangle \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}\langle \boldsymbol{A}(1,:), \boldsymbol{B}(1,:) \rangle + a_{12}\langle \boldsymbol{A}(1,:), \boldsymbol{B}(2,:) \rangle + \cdots + a_{1K}\langle \boldsymbol{A}(1,:), \boldsymbol{B}(K,:) \rangle \\ a_{21}\langle \boldsymbol{A}(1,:), \boldsymbol{B}(1,:) \rangle + a_{22}\langle \boldsymbol{A}(1,:), \boldsymbol{B}(2,:) \rangle + \cdots + a_{2K}\langle \boldsymbol{A}(1,:), \boldsymbol{B}(K,:) \rangle \\ \vdots \\ a_{I1}\langle \boldsymbol{A}(1,:), \boldsymbol{B}(1,:) \rangle + a_{I2}\langle \boldsymbol{A}(1,:), \boldsymbol{B}(2,:) \rangle + \cdots + a_{IK}\langle \boldsymbol{A}(1,:), \boldsymbol{B}(K,:) \rangle \\ a_{11}\langle \boldsymbol{A}(2,:), \boldsymbol{B}(1,:) \rangle + a_{12}\langle \boldsymbol{A}(2,:), \boldsymbol{B}(2,:) \rangle + \cdots + a_{1K}\langle \boldsymbol{A}(2,:), \boldsymbol{B}(K,:) \rangle \\ \vdots \\ a_{I1}\langle \boldsymbol{A}(I,:), \boldsymbol{B}(1,:) \rangle + a_{I2}\langle \boldsymbol{A}(I,:), \boldsymbol{B}(2,:) \rangle + \cdots + a_{IK}\langle \boldsymbol{A}(I,:), \boldsymbol{B}(K,:) \rangle \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}\boldsymbol{A} & a_{12}\boldsymbol{A} & \cdots \\ a_{21}\boldsymbol{A} & a_{22}\boldsymbol{A} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \boldsymbol{B}(:,1) \\ \boldsymbol{B}(:,2) \\ \vdots \\ \boldsymbol{B}(:,K) \end{pmatrix} = (\boldsymbol{A} \otimes \boldsymbol{A})\,\mathrm{vec}(\boldsymbol{B})$$