The main reasoning problem considered in this chapter is given a set of unit equations $E$ and an additional equation $s \approx t$, does $E \models s \approx t$ hold?

As usual, all variables are implicitly universally quantified. The idea is to turn the equations $E$ into a convergent term rewrite system (TRS) $R$ such that the above problem can be solved by checking identity of the respective normal forms: $s \downarrow_R = t \downarrow_R$.

Showing $E \models s \approx t$ is as difficult as proving validity of any first-order formula, see the section on complexity.
4.0.1 Definition (Equivalence Relation, Congruence Relation)

An *equivalence* relation $\sim$ on a term set $T(\Sigma, \mathcal{X})$ is a reflexive, transitive, symmetric binary relation on $T(\Sigma, \mathcal{X})$ such that if $s \sim t$ then $\text{sort}(s) = \text{sort}(t)$.

Two terms $s$ and $t$ are called *equivalent*, if $s \sim t$.

An equivalence $\sim$ is called a *congruence* if $s \sim t$ implies $u[s] \sim u[t]$, for all terms $s, t, u \in T(\Sigma, \mathcal{X})$. Given a term $t \in T(\Sigma, \mathcal{X})$, the set of all terms equivalent to $t$ is called the *equivalence class of $t$ by $\sim$*, denoted by

$$[t]_\sim := \{ t' \in T(\Sigma, \mathcal{X}) \mid t' \sim t \}.$$
If the matter of discussion does not depend on a particular equivalence relation or it is unambiguously known from the context, \([t]\) is used instead of \([t] \sim\). The above definition is equivalent to Definition 3.2.3.

The set of all equivalence classes in \(T(\Sigma, \mathcal{X})\) defined by the equivalence relation is called a *quotient by \(\sim\)*, denoted by 
\[T(\Sigma, \mathcal{X})|_{\sim} := \{[t] \mid t \in T(\Sigma, \mathcal{X})\}.
\]

Let \(E\) be a set of equations then \(\sim_E\) denotes the smallest congruence relation “containing” \(E\), that is, 
\[(l \approx r) \in E\text{ implies } l \sim_E r.\]

The equivalence class \([t]_{\sim_E}\) of a term \(t\) by the equivalence (congruence) \(\sim_E\) is usually denoted, for short, by \([t]_E\). Likewise, 
\[T(\Sigma, \mathcal{X})|_E\] is used for the quotient 
\[T(\Sigma, \mathcal{X})|_{\sim_E}\] of \(T(\Sigma, \mathcal{X})\) by the equivalence (congruence) \(\sim_E\).
4.1.1 Definition (Rewrite Rule, Term Rewrite System)

A \textit{rewrite rule} is an equation \( l \approx r \) between two terms \( l \) and \( r \) so that \( l \) is not a variable and \( \text{vars}(l) \supseteq \text{vars}(r) \). A \textit{term rewrite system} \( R \), or a TRS for short, is a set of rewrite rules.

4.1.2 Definition (Rewrite Relation)

Let \( E \) be a set of (implicitly universally quantified) equations, i.e., unit clauses containing exactly one positive equation. The \textit{rewrite relation} \( \rightarrow_E \subseteq T(\Sigma, \mathcal{X}) \times T(\Sigma, \mathcal{X}) \) is defined by

\[
s \rightarrow_E t \quad \text{iff} \quad \begin{align*}
\exists (l \approx r) \in E, p \in \text{pos}(s), \\
\text{and matcher } \sigma, \text{ so that } s|_p &= l\sigma \text{ and } t = s[r\sigma]_p.
\end{align*}
\]
Note that in particular for any equation \( l \approx r \in E \) it holds \( l \rightarrow_E r \), so the equation can also be written \( l \rightarrow r \in E \).

Often \( s = t \downarrow_R \) is written to denote that \( s \) is a normal form of \( t \) with respect to the rewrite relation \( \rightarrow_R \). Notions \( \rightarrow_R^0, \rightarrow_R^+, \rightarrow_R^*, \leftrightarrow_R^* \), etc. are defined accordingly, see Section 1.6.
An instance of the left-hand side of an equation is called a *redex* (reducible expression). *Contracting* a redex means replacing it with the corresponding instance of the right-hand side of the rule.

A term rewrite system $R$ is called *convergent* if the rewrite relation $\rightarrow_R$ is confluent and terminating. A set of equations $E$ or a TRS $R$ is terminating if the rewrite relation $\rightarrow_E$ or $\rightarrow_R$ has this property. Furthermore, if $E$ is terminating then it is a TRS.

A rewrite system is called *right-reduced* if for all rewrite rules $l \rightarrow r$ in $R$, the term $r$ is irreducible by $R$. A rewrite system $R$ is called *left-reduced* if for all rewrite rules $l \rightarrow r$ in $R$, the term $l$ is irreducible by $R \setminus \{l \rightarrow r\}$. A rewrite system is called *reduced* if it is left- and right-reduced.
4.1.3 Lemma (Left-Reduced TRS)
Left-reduced terminating rewrite systems are convergent. Convergent rewrite systems define unique normal forms.

4.1.4 Lemma (TRS Termination)
A rewrite system $R$ terminates iff there exists a reduction ordering $\succ$ so that $l \succ r$, for each rule $l \rightarrow r$ in $R$. 
Let $E$ be a set of universally quantified equations. A model $\mathcal{A}$ of $E$ is also called an $E$-algebra. If $E \models \forall \vec{x}(s \approx t)$, i.e., $\forall \vec{x}(s \approx t)$ is valid in all $E$-algebras, this is also denoted with $s \approx^E t$. The goal is to use the rewrite relation $\rightarrow_E$ to express the semantic consequence relation syntactically: $s \approx^E t$ if and only if $s \leftrightarrow^*_E t$.

Let $E$ be a set of (well-sorted) equations over $T(\Sigma, \mathcal{X})$ where all variables are implicitly universally quantified. The following inference system allows to derive consequences of $E$: 
Reflexivity \[ E \Rightarrow_E E \cup \{ t \approx t \} \]

Symmetry \[ E \cup \{ t \approx t' \} \Rightarrow_E E \cup \{ t \approx t' \} \cup \{ t' \approx t \} \]

Transitivity \[ E \cup \{ t \approx t', t' \approx t'' \} \Rightarrow_E E \cup \{ t \approx t', t' \approx t'' \} \cup \{ t \approx t'' \} \]
Congruence \[ E \cup \{ t_1 \approx t'_1, \ldots, t_n \approx t'_n \} \Rightarrow_E \]
\[ E \cup \{ t_1 \approx t'_1, \ldots, t_n \approx t'_n \} \cup \{ f(t_1, \ldots, t_n) \approx f(t'_1, \ldots, t'_n) \} \]
for any function \( f : \text{sort}(t_1) \times \ldots \times \text{sort}(t_n) \to S \) for some \( S \)

Instance \[ E \cup \{ t \approx t' \} \Rightarrow_E E \cup \{ t \approx t' \} \cup \{ t\sigma \approx t'\sigma \} \]
for any well-sorted substitution \( \sigma \)
4.1.5 Lemma (Equivalence of $\leftrightarrow^*_E$ and $\Rightarrow^*_E$)

The following properties are equivalent:

1. $s \leftrightarrow^*_E t$

2. $E \Rightarrow^*_E s \approx t$ is derivable.

where $E \Rightarrow^*_E s \approx t$ is an abbreviation for $E \Rightarrow^*_E E'$ and $s \approx t \in E'$. 
4.1.6 Corollary (Convergence of $E$)

If a set of equations $E$ is convergent then $s \approx_E t$ if and only if $s \leftrightarrow^* t$ if and only if $s \downarrow_E = t \downarrow_E$.

4.1.7 Corollary (Decidability of $\approx_E$)

If a set of equations $E$ is finite and convergent then $\approx_E$ is decidable.
The above Lemma 4.1.5 shows equivalence of the syntactically defined relations $\leftrightarrow^*_E$ and $\Rightarrow^*_E$. What is missing, in analogy to Herbrand’s theorem for first-order logic without equality Theorem 3.5.5, is a semantic characterization of the relations by a particular algebra.

4.1.8 Definition (Quotient Algebra)

For sets of unit equations this is a quotient algebra: Let $X$ be a set of variables. For $t \in T(\Sigma, \mathcal{X})$ let

$$[t] = \{ t' \in T(\Sigma, \mathcal{X}) \mid E \Rightarrow^*_E t \approx t' \}$$

be the congruence class of $t$. Define a $\Sigma$-algebra $\mathcal{I}_E$, called the quotient algebra, technically $T(\Sigma, \mathcal{X})/E$, as follows: $S^{\mathcal{I}_E} = \{ [t] \mid t \in T_S(\Sigma, \mathcal{X}) \}$ for all sorts $S$ and $f^{\mathcal{I}_E}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$ for $f : \text{sort}(t_1) \times \ldots \times \text{sort}(t_n) \rightarrow T \in \Omega$ for some sort $T$. 
4.1.9 Lemma ($\mathcal{I}_E$ is an $E$-algebra)

$\mathcal{I}_E = T(\Sigma, \mathcal{X})/E$ is an $E$-algebra.

4.1.10 Lemma ($\Rightarrow_E$ is complete)

Let $\mathcal{X}$ be a countably infinite set of variables; let $s, t \in T_S(\Sigma, \mathcal{X})$. If $\mathcal{I}_E \models \forall \vec{x}(s \approx t)$, then $E \Rightarrow^*_E s \approx t$ is derivable.