Branch and Bound for LIA

Idea: given a set of LIA inequations N, find a solution by relaxation to LRA and case split with respect to an LRA solution that is not in the integers.

6.2.13 Lemma (LIA Satisfiability is NP-Complete)

Let N be a conjunction of linear arithmetic constraints then LIA $\models \exists x_1, \ldots, x_n . N$ is NP-complete.

Proof.

NP-Membership: If N contains n variables and a is the absulte value of the largest coefficients, then all variables can be bound to $-n(|N|a)^{2|N|+1} \leq \beta(x_i) \leq n(|N|a)^{2|N|+1}$.

NP-Hardness: By coding 3-SAT. □

The simple LIA branch and bound calculus is very similar to DPLL, Section 2.8. A LIABB problem state is a pair $(M; N)$ where M a sequence of partly annotated simple bounds $x_i \leq d$, $d \in \mathbb{Z}$, and N is a set of inequations, $\text{vars}(N) = \{x_1, \ldots, x_n\}$. Let a be the maximal absolute value of a coefficient in N, $c = n(|N|a)^{2|N|+1}$, then the following LIABB states can be distinguished:

$(B; N)$	is the start state for N, where $B = -c \leq x_1, x_1 \leq c, \ldots, -c \leq x_n, x_n \leq c$.
$(M; N)$	is a final state, if there is a unique β, $\text{LIA}(\beta) \models M \wedge N$
$(M; N)$	is a final state, if there is no β, $\text{LIA}(\beta) \models N$

Given a state (M, N), a simple bound $x \circ d$, $d \in \mathbb{Z}$, is called *undefined* in M, if there exists a valuation β, $\mathrm{LIA}(\beta) \models M$ and $\mathrm{LIA}(\beta) \not\models x \circ d$. The rules Propagate, Decide, and Backtrack constitute the LIABB calculus.

Propagate $(M; N) \Rightarrow_{\text{LIABB}} (M, x \circ d; N)$
provided there is a valuation β, $\text{LRA}(\beta) \models M \wedge N$, $\text{LIA} \models \forall x_1, \ldots, x_n.[(M \wedge N) \rightarrow x \circ d]$, $d \in \mathbb{Z}$, and $x \circ d$ is undefined in M

Decide $(M; N) \Rightarrow_{\text{LIABB}} (M, x \circ e^d; N)$
provided $x \circ e$ is undefined in M, $\text{LRA}(\beta) \models M \wedge N$, $\beta(x) = d$ and either ($\circ = \leq$ and $e = \lfloor d \rfloor$) or ($\circ = \geq$ and $e = \lceil d \rceil$)

Backtrack $(M_1, x \circ_1 e_1^d, M_2; N) \Rightarrow_{\text{LIABB}} (M_1, x \circ_2 e_2; N)$
provided there is no valuation β, $\text{LRA}(\beta) \models (M_1 \wedge x \circ_1 e_1 \wedge M_2 \wedge N)$ and there is no $y \circ' e'^{d'}$ in M_2 and if $\circ_1 = \leq$, then $\circ_2 = \geq$ and $e_2 = \lceil d \rceil$; if $\circ_1 = \geq$, then $\circ_2 = \leq$ and $e_2 = \lfloor d \rfloor$

6.2.14 Lemma (LIABB Propagate and Decide)

Let $(B, N) \Rightarrow^*_{\text{LIABB}} (M, N)$ be a LIABB derivation. Then from (M, N) there only finitely many applications of Propagate and Decide possible.

6.2.15 Theorem (LIABB Terminates)

Any derivation $(B, N) \Rightarrow^*_{\text{LIABB}} \ldots$ is finite.