Normal Forms

Definition (CNF, DNF)

A formula is in *conjunctive normal form (CNF)* or *clause normal form* if it is a conjunction of disjunctions of literals, or in other words, a conjunction of clauses.

A formula is in *disjunctive normal form (DNF)*, if it is a disjunction of conjunctions of literals.
Checking the validity of CNF formulas or the unsatisfiability of DNF formulas is easy:

(i) a formula in CNF is valid, if and only if each of its disjunctions contains a pair of complementary literals $P$ and $\neg P$,

(ii) conversely, a formula in DNF is unsatisfiable, if and only if each of its conjunctions contains a pair of complementary literals $P$ and $\neg P$
Basic CNF Transformation

**ElimEquiv** \[ \chi[(\phi \leftrightarrow \psi)]_\rho \Rightarrow \text{BCNF} \quad \chi[\left(\phi \rightarrow \psi\right) \land \left(\psi \rightarrow \phi\right)]_\rho \]

**ElimImp** \[ \chi[(\phi \rightarrow \psi)]_\rho \Rightarrow \text{BCNF} \quad \chi[\left(\neg \phi \lor \psi\right)]_\rho \]

**PushNeg1** \[ \chi[\neg(\phi \lor \psi)]_\rho \Rightarrow \text{BCNF} \quad \chi[\left(\neg \phi \land \neg \psi\right)]_\rho \]

**PushNeg2** \[ \chi[\neg(\phi \land \psi)]_\rho \Rightarrow \text{BCNF} \quad \chi[\left(\neg \phi \lor \neg \psi\right)]_\rho \]

**PushNeg3** \[ \chi[\neg\neg \phi]_\rho \Rightarrow \text{BCNF} \quad \chi[\phi]_\rho \]

**PushDisj** \[ \chi[\left(\phi_1 \land \phi_2\right) \lor \psi]_\rho \Rightarrow \text{BCNF} \quad \chi[\left(\phi_1 \lor \psi\right) \land \left(\phi_2 \lor \psi\right)]_\rho \]

**ElimTB1** \[ \chi[\left(\phi \land \top\right)]_\rho \Rightarrow \text{BCNF} \quad \chi[\phi]_\rho \]

**ElimTB2** \[ \chi[\left(\phi \land \bot\right)]_\rho \Rightarrow \text{BCNF} \quad \chi[\bot]_\rho \]

**ElimTB3** \[ \chi[\left(\phi \lor \top\right)]_\rho \Rightarrow \text{BCNF} \quad \chi[\top]_\rho \]

**ElimTB4** \[ \chi[\left(\phi \lor \bot\right)]_\rho \Rightarrow \text{BCNF} \quad \chi[\phi]_\rho \]

**ElimTB5** \[ \chi[\neg \bot]_\rho \Rightarrow \text{BCNF} \quad \chi[\top]_\rho \]

**ElimTB6** \[ \chi[\neg \top]_\rho \Rightarrow \text{BCNF} \quad \chi[\bot]_\rho \]
Basic CNF Algorithm

1 **Algorithm:** \( \text{bcnf}(\phi) \)

**Input**: A propositional formula \( \phi \).

**Output**: A propositional formula \( \psi \) equivalent to \( \phi \) in CNF.

2 while rule \( \text{ElimEquiv}(\phi) \) do ;
3 while rule \( \text{ElimImp}(\phi) \) do ;
4 while rule \( \text{ElimTB1}(\phi), \ldots, \text{ElimTB6}(\phi) \) do ;
5 while rule \( \text{PushNeg1}(\phi), \ldots, \text{PushNeg3}(\phi) \) do ;
6 while rule \( \text{PushDisj}(\phi) \) do ;
7 return \( \phi \);
Advanced CNF Algorithm

For the formula

\[ P_1 \leftrightarrow (P_2 \leftrightarrow (P_3 \leftrightarrow (\ldots (P_{n-1} \leftrightarrow P_n)\ldots))) \]

the basic CNF algorithm generates a CNF with \(2^{n-1}\) clauses.
2.5.4 Proposition (Renaming Variables)

Let $P$ be a propositional variable not occurring in $\psi[\phi]_p$.

1. If $\text{pol}(\psi, p) = 1$, then $\psi[\phi]_p$ is satisfiable if and only if $\psi[P]_p \land (P \rightarrow \phi)$ is satisfiable.

2. If $\text{pol}(\psi, p) = -1$, then $\psi[\phi]_p$ is satisfiable if and only if $\psi[P]_p \land (\phi \rightarrow P)$ is satisfiable.

3. If $\text{pol}(\psi, p) = 0$, then $\psi[\phi]_p$ is satisfiable if and only if $\psi[P]_p \land (P \leftrightarrow \phi)$ is satisfiable.
Renaming

**SimpleRenaming** \( \phi \Rightarrow_{\text{SimpRen}} \phi[P_1]p_1[P_2]p_2 \ldots [P_n]p_n \land \text{def}(\phi, p_1, P_1) \land \ldots \land \text{def}(\phi[P_1]p_1[P_2]p_2 \ldots [P_{n-1}]p_{n-1}, p_n, P_n) \)

provided \( \{p_1, \ldots, p_n\} \subset \text{pos}(\phi) \) and for all \( i, i+j \) either \( p_i \parallel p_{i+j} \) or \( p_i > p_{i+j} \) and the \( P_i \) are different and new to \( \phi \).

Simple choice: choose \( \{p_1, \ldots, p_n\} \) to be all non-literal and non-negation positions of \( \phi \).
Renaming Definition

\[
def(\psi, \rho, P) := \begin{cases} 
(P \rightarrow \psi|\rho) & \text{if } \text{pol}(\psi, \rho) = 1 \\
(\psi|\rho \rightarrow P) & \text{if } \text{pol}(\psi, \rho) = -1 \\
(P \leftrightarrow \psi|\rho) & \text{if } \text{pol}(\psi, \rho) = 0
\end{cases}
\]
Obvious Positions

A smaller set of positions from $\phi$, called *obvious positions*, is still preventing the explosion and given by the rules:

(i) $p$ is an obvious position if $\phi|_p$ is an equivalence and there is a position $q < p$ such that $\phi|_q$ is either an equivalence or disjunctive in $\phi$ or

(ii) $pq$ is an obvious position if $\phi|_{pq}$ is a conjunctive formula in $\phi$, $\phi|_p$ is a disjunctive formula in $\phi$ and for all positions $r$ with $p < r < pq$ the formula $\phi|_r$ is not a conjunctive formula.

A formula $\phi|_p$ is conjunctive in $\phi$ if $\phi|_p$ is a conjunction and $\text{pol}(\phi, p) \in \{0, 1\}$ or $\phi|_p$ is a disjunction or implication and $\text{pol}(\phi, p) \in \{0, -1\}$.

Analogously, a formula $\phi|_p$ is disjunctive in $\phi$ if $\phi|_p$ is a disjunction or implication and $\text{pol}(\phi, p) \in \{0, 1\}$ or $\phi|_p$ is a conjunction and $\text{pol}(\phi, p) \in \{0, -1\}$.
Polarity Dependent Equivalence
Elimination

**ElimEquiv1** \[ \chi[(\phi \leftrightarrow \psi)]_p \Rightarrow_{\text{ACNF}} \chi[(\phi \to \psi) \land (\psi \to \phi)]_p \]
provided \( \text{pol}(\chi, p) \in \{0, 1\} \)

**ElimEquiv2** \[ \chi[(\phi \leftrightarrow \psi)]_p \Rightarrow_{\text{ACNF}} \chi[(\phi \land \psi) \lor (\neg \phi \land \neg \psi)]_p \]
provided \( \text{pol}(\chi, p) = -1 \)
Extra $\top, \bot$ Elimination Rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>ElimTB7</td>
<td>$\chi[\phi \rightarrow \bot] \Rightarrow \text{ACNF} \chi[\neg \phi] \rho$</td>
<td></td>
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<tr>
<td>ElimTB8</td>
<td>$\chi[\bot \rightarrow \phi] \Rightarrow \text{ACNF} \chi[\top] \rho$</td>
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<tr>
<td>ElimTB9</td>
<td>$\chi[\phi \rightarrow \top] \Rightarrow \text{ACNF} \chi[\top] \rho$</td>
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<tr>
<td>ElimTB10</td>
<td>$\chi[\top \rightarrow \phi] \Rightarrow \text{ACNF} \chi[\phi] \rho$</td>
<td></td>
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<tr>
<td>ElimTB11</td>
<td>$\chi[\phi \leftrightarrow \bot] \Rightarrow \text{ACNF} \chi[\neg \phi] \rho$</td>
<td></td>
</tr>
<tr>
<td>ElimTB12</td>
<td>$\chi[\phi \leftrightarrow \top] \Rightarrow \text{ACNF} \chi[\phi] \rho$</td>
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</tbody>
</table>

where the two rules ElimTB11, ElimTB12 for equivalences are applied with respect to commutativity of $\leftrightarrow$. 
Advanced CNF Algorithm

1 Algorithm: \texttt{acnf}(\phi)

Input  : A formula \phi.
Output: A formula \psi in CNF satisfiability preserving to \phi.

2 \texttt{while rule}\ (\texttt{ElimTB1}(\phi), \ldots, \texttt{ElimTB12}(\phi)) \texttt{do} ;

3 \texttt{SimpleRenaming}(\phi) on obvious positions;

4 \texttt{while rule}\ (\texttt{ElimEquiv1}(\phi), \texttt{ElimEquiv2}(\phi)) \texttt{do} ;

5 \texttt{while rule}\ (\texttt{ElimImp}(\phi)) \texttt{do} ;

6 \texttt{while rule}\ (\texttt{PushNeg1}(\phi), \ldots, \texttt{PushNeg3}(\phi)) \texttt{do} ;

7 \texttt{while rule}\ (\texttt{PushDisj}(\phi)) \texttt{do} ;

8 return \phi;