Propositional Superposition

Propositional Superposition refines the propositional resolution calculus by

(i) ordering and selection restrictions on inferences,

(ii) an abstract redundancy notion,

(iii) the notion of a partial model, based on the ordering for inference guidance

(iv) a *saturation* concept.

Important: No implicit Condensation of literals!

2.7.1 Definition (Clause Ordering)

Let $\prec$ be a total strict ordering on Σ.

Then $\prec$ can be lifted to a total ordering on literals by $\prec \subseteq \prec_L$ and $P \prec_L \neg P$ and $\neg P \prec_L Q$, $\neg P \prec_L \neg Q$ for all $P \prec Q$.

The ordering $\prec_L$ can be lifted to a total ordering on clauses $\prec_C$ by considering the multiset extension of $\prec_L$ for clauses.

2.7.2 Proposition (Properties of the Clause Ordering)

(i) The orderings on literals and clauses are total and well-founded.
(ii) Let C and D be clauses with $P = \mathrm{atom}(\max(C))$, $Q = \mathrm{atom}(\max(D))$, where $\max(C)$ denotes the maximal literal in C.

(i) If $Q \prec_L P$ then $D \prec_C C$.
(ii) If $P = Q$, P occurs negatively in C but only positively in D, then $D \prec_C C$.

Eventually, I overload $\prec$ with $\prec_L$ and $\prec_C$.

For a clause set N, I define $N^{\prec C} = \{D \in N \mid D \prec C\}$.

Definition (Abstract Redundancy)

A clause C is *redundant* with respect to a clause set N if $N^{\prec C} \models C$.

2.7.5 Definition (Selection Function)

The selection function sel maps clauses to one of its negative literals or $\perp$.

If $\text{sel}(C) = \neg P$ then $\neg P$ is called *selected* in C.

If $\text{sel}(C) = \perp$ then no literal in C is *selected*.

2.7.6 Definition (Partial Model Construction)

Given a clause set N and an ordering $\prec$ we can construct a (partial) Herbrand model $N_\mathcal{I}$ for N inductively as follows:

$$
\begin{aligned}
N_C &:= \bigcup_{D \prec C} \delta_D \\
\delta_D &:= \begin{cases} \{P\} & \text{if } D = D' \lor P,\ P \text{ strictly maximal, no literal} \\ & \text{selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases} \\
N_\mathcal{I} &:= \bigcup_{C \in N} \delta_C
\end{aligned}
$$

Clauses C with $\delta_C \neq \emptyset$ are called *productive*.

2.7.7 Proposition (Model Construction Properties)

Some properties of the partial model construction.

(i) For every D with $(C \vee \neg P) \prec D$ we have $\delta_D \neq \{P\}$.

(ii) If $\delta_C = \{P\}$ then $N_C \cup \delta_C \models C$.

(iii) If $N_C \models D$ and $D \prec C$ then for all C' with $C \prec C'$ we have $N_{C'} \models D$ and in particular $N_{\mathcal{I}} \models D$.

(iv) There is no clause C with $P \vee P \prec C$ such that $\delta_C = \{P\}$.

Superposition Inference Rules

Superposition Left $(N \uplus \{C_1 \vee P, C_2 \vee \neg P\}) \Rightarrow_{\text{SUP}}$
$(N \cup \{C_1 \vee P, C_2 \vee \neg P\} \cup \{C_1 \vee C_2\})$

where (i) P is strictly maximal in $C_1 \vee P$ (ii) no literal in $C_1 \vee P$ is selected (iii) $\neg P$ is maximal and no literal selected in $C_2 \vee \neg P$, or $\neg P$ is selected in $C_2 \vee \neg P$

Factoring $(N \uplus \{C \vee P \vee P\}) \Rightarrow_{\text{SUP}}$
$(N \cup \{C \vee P \vee P\} \cup \{C \vee P\})$

where (i) P is maximal in $C \vee P \vee P$ (ii) no literal is selected in $C \vee P \vee P$

2.7.8 Definition (Saturation)

A set N of clauses is called *saturated up to redundancy*, if any inference from non-redundant clauses in N yields a redundant clause with respect to N or is already contained in N.

Superposition Reduction Rules

Subsumption $(N \uplus \{C_1, C_2\}) \Rightarrow_{\text{SUP}} (N \cup \{C_1\})$
provided $C_1 \subset C_2$

Tautology Deletion $(N \uplus \{C \lor P \lor \neg P\}) \Rightarrow_{\text{SUP}} (N)$

Condensation $(N \uplus \{C_1 \lor L \lor L\}) \Rightarrow_{\text{SUP}} (N \cup \{C_1 \lor L\})$

Subsumption Resolution $(N \uplus \{C_1 \lor L, C_2 \lor \text{comp}(L)\}) \Rightarrow_{\text{SUP}}$
$(N \cup \{C_1 \lor L, C_2\})$
where $C_1 \subseteq C_2$

2.7.9 Proposition (Reduction Rules)

All clauses removed by Subsumption, Tautology Deletion, Condensation and Subsumption Resolution are redundant with respect to the kept or added clauses.

2.7.10 Corollary (Soundness)

Superposition is sound.

2.7.11 Theorem (Completeness)

If N is saturated up to redundancy and $\bot \notin N$ then N is satisfiable and $N_{\mathcal{I}} \models N$.