Unification

3.7.1 Definition (Unifier)

Two terms $s$ and $t$ of the same sort are said to be unifiable if there exists a well-sorted substitution $\sigma$ so that $s\sigma = t\sigma$, the substitution $\sigma$ is then called a well-sorted unifier of $s$ and $t$.

The unifier $\sigma$ is called most general unifier, written $\sigma = \text{mgu}(s, t)$, if any other well-sorted unifier $\tau$ of $s$ and $t$ it can be represented as $\tau = \sigma\tau'$, for some well-sorted substitution $\tau'$. 
A state of the naive standard unification calculus is a set of equations \( E \) or \( \perp \), where \( \perp \) denotes that no unifier exists. The set \( E \) is also called a unification problem.

The start state for checking whether two terms \( s, t \), \( \text{sort}(s) = \text{sort}(t) \), (or two non-equational atoms \( A, B \)) are unifiable is the set \( E = \{ s = t \} \) (or \( E = \{ A = B \} \)). A variable \( x \) is solved in \( E \) if \( E = \{ x = t \} \uplus E' \), \( x \notin \text{vars}(t) \) and \( x \notin \text{vars}(E) \).

A variable \( x \in \text{vars}(E) \) is called solved in \( E \) if \( E = E' \uplus \{ x = t \} \) and \( x \notin \text{vars}(t) \) and \( x \notin \text{vars}(E') \).
Standard (naive) Unification

**Tautology**

\[
E \uplus \{ t = t \} \Rightarrow_{SU} E
\]

**Decomposition**

\[
E \uplus \{ f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n) \} \Rightarrow_{SU} E \cup \{ s_1 = t_1, \ldots, s_n = t_n \}
\]

**Clash**

\[
E \uplus \{ f(s_1, \ldots, s_n) = g(s_1, \ldots, s_m) \} \Rightarrow_{SU} \bot
\]
if \( f \neq g \)
Substitution

\[ E \cup \{ x = t \} \Rightarrow_{SU} E \{ x \mapsto t \} \cup \{ x = t \} \]

if \( x \in \text{vars}(E) \) and \( x \not\in \text{vars}(t) \)

Occurs Check

\[ E \cup \{ x = t \} \Rightarrow_{SU} \bot \]

if \( x \neq t \) and \( x \in \text{vars}(t) \)

Orient

\[ E \cup \{ t = x \} \Rightarrow_{SU} E \cup \{ x = t \} \]

if \( t \not\in X \)
3.7.2 Theorem (Soundness, Completeness and Termination of $\Rightarrow_{SU}$)

If $s, t$ are two terms with $\text{sort}(s) = \text{sort}(t)$ then

1. if $\{ s = t \} \Rightarrow_{SU}^* E$ then any equation $(s' = t') \in E$ is well-sorted, i.e., $\text{sort}(s') = \text{sort}(t')$.

2. $\Rightarrow_{SU}$ terminates on $\{ s = t \}$.

3. if $\{ s = t \} \Rightarrow_{SU}^* E$ then $\sigma$ is a unifier (mgu) of $E$ iff $\sigma$ is a unifier (mgu) of $\{ s = t \}$.

4. if $\{ s = t \} \Rightarrow_{SU}^* \perp$ then $s$ and $t$ are not unifiable.

5. if $\{ s = t \} \Rightarrow_{SU}^* \{ x_1 = t_1, \ldots, x_n = t_n \}$ and this is a normal form, then $\{ x_1 \mapsto t_1, \ldots, x_n \mapsto t_n \}$ is an mgu of $s, t$. 
Size of Unification Problems

Any normal form of the unification problem $E$ given by

$$\{ f(x_1, g(x_1, x_1), x_3, \ldots, g(x_n, x_n)) = f(g(x_0, x_0), x_2, g(x_2, x_2), \ldots, x_{n+1}) \}$$

with respect to $\Rightarrow_{SU}$ is exponentially larger than $E$. 
Polynomial Unification

The second calculus, polynomial unification, prevents the problem of exponential growth by introducing an implicit representation for the mgu.

For this calculus the size of a normal form is always polynomial in the size of the input unification problem.
Tautology \[ E \cup \{ t = t \} \Rightarrow_{PU} E \]

Decomposition \[ E \cup \{ f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n) \} \Rightarrow_{PU} E \cup \{ s_1 = t_1, \ldots, s_n = t_n \} \]

Clash \[ E \cup \{ f(t_1, \ldots, t_n) = g(s_1, \ldots, s_m) \} \Rightarrow_{PU} \bot \]
if \( f \neq g \)
Occurs Check \[ E \uplus \{ x = t \} \Rightarrow_{PU} \bot \]
if \( x \neq t \) and \( x \in \text{vars}(t) \)

Orient \[ E \uplus \{ t = x \} \Rightarrow_{PU} E \uplus \{ x = t \} \]
if \( t \notin X \)

Substitution \[ E \uplus \{ x = y \} \Rightarrow_{PU} E \{ x \mapsto y \} \uplus \{ x = y \} \]
if \( x \in \text{vars}(E) \) and \( x \neq y \)
Cycle \[ E \uplus \{ x_1 = t_1, \ldots, x_n = t_n \} \Rightarrow_{PU} \bot \]
if there are positions \( p_i \) with \( t_i|_{p_i} = x_{i+1}, t_n|_{p_n} = x_1 \) and some \( p_i \neq \epsilon \)

Merge \[ E \uplus \{ x = t, x = s \} \Rightarrow_{PU} E \uplus \{ x = t, t = s \} \]
if \( t, s \not\in \mathcal{X} \) and \( |t| \leq |s| \)
3.7.4 Theorem (Soundness, Completeness and Termination of $\Rightarrow_{PU}$)

If $s$, $t$ are two terms with $\text{sort}(s) = \text{sort}(t)$ then

1. if $\{s = t\} \Rightarrow^*_{PU} E$ then any equation $(s' = t') \in E$ is well-sorted, i.e., $\text{sort}(s') = \text{sort}(t')$.

2. $\Rightarrow_{PU}$ terminates on $\{s = t\}$.

3. if $\{s = t\} \Rightarrow^*_{PU} E$ then $\sigma$ is a unifier (mgu) of $E$ iff $\sigma$ is a unifier (mgu) of $\{s = t\}$.

4. if $\{s = t\} \Rightarrow^*_{PU} \perp$ then $s$ and $t$ are not unifiable.
3.7.5 Theorem (Normal Forms Generated by $\Rightarrow^*_{PU}$)

Let $\{s = t\} \Rightarrow^*_{PU} \{x_1 = t_1, \ldots, x_n = t_n\}$ be a normal form. Then

1. $x_i \neq x_j$ for all $i \neq j$ and without loss of generality $x_i \notin \text{vars}(t_{i+k})$ for all $i, k$, $1 \leq i < n$, $i + k \leq n$.

2. the substitution $\{x_1 \mapsto t_1\}\{x_2 \mapsto t_2\} \ldots \{x_n \mapsto t_n\}$ is an mgu of $s = t$. 