Congruence Closure

An equational clause

$$\forall \vec{x} (t_1 \approx s_1 \lor \ldots \lor t_n \approx s_n \lor l_1 \not\approx r_1 \lor \ldots \lor l_k \not\approx r_k)$$

is valid iff

$$\exists \vec{x} (t_1 \not\approx s_1 \land \ldots \land t_n \not\approx s_n \land l_1 \approx r_1 \land \ldots \land l_k \approx r_k)$$

is unsatisfiable iff the Skolemized (ground!) formula

$$(t_1 \not\approx s_1 \land \ldots \land t_n \not\approx s_n \land l_1 \approx r_1 \land \ldots \land l_k \approx r_k)\{\vec{x} \mapsto \vec{c}\}$$

is unsatisfiable iff the formula

$$(t_1 \approx s_1 \lor \ldots \lor t_n \approx s_n \lor l_1 \not\approx r_1 \lor \ldots \lor l_k \not\approx r_k)\{\vec{x} \mapsto \vec{c}\}$$

is valid.
Flattening

\[ E = l_1 \approx r_1 \land \ldots \land l_n \approx r_n \]

**Flattening**

\[ E[f(t_1, \ldots, t_n)]_{p_1, \ldots, p_k} \Rightarrow_{CCF} \]

\[ E[c/p_1, \ldots, p_k] \land f(t_1, \ldots, t_n) \approx c \]

provided all \( t_i \) are constants, the \( p_j \) are all positions in \( E \) of \( f(t_1, \ldots, t_n) \), \( |p_k| > 2 \) for some \( k \), or, \( p_k = n.2 \) and \( E|_{m.1} \) is not a constant for some \( n \), and \( c \) is fresh
As a result: only two kinds of equations left.
Term equations: $f(c_{i_1}, \ldots, c_{i_n}) \approx c_{i_0}$
Constant equations: $c_i \approx c_j$. 
Congruence Closure

The congruence closure algorithm is presented as a set of abstract rewrite rules operating on a pair of equations $E$ and a set of rules $R$, $(E; R)$, similar to Knuth-Bendix completion, Section 4.4.

$(E_0; R_0) \Rightarrow_{CC} (E_1; R_1) \Rightarrow_{CC} (E_2; R_2) \Rightarrow_{CC} \ldots$

At the beginning, $E = E_0$ is a set of constant equations and $R = R_0$ is the set of term equations oriented from left-to-right. At termination, $E$ is empty and $R$ contains the result.
Simplify \((E \cup \{c \approx c\}; R \cup \{c \rightarrow c''\}) \Rightarrow_{CC} (E \cup \{c'' \approx c\}; R \cup \{c \rightarrow c''\})\)

Delete \((E \cup \{c \approx c\}; R) \Rightarrow_{CC} (E; R)\)

Orient \((E \cup \{c \approx c\}; R) \Rightarrow_{CC} (E; R \cup \{c \rightarrow c\})\) if \(c \succ c'\)
Decidable Logics

For rule Deduce, \( t \) is either a term of the form \( f(c_1, \ldots, c_n) \) or a constant \( c_i \).

For rule Collapse, \( t \) is always of the form \( f(c_1, \ldots, c_n) \).