Congruence Closure

An equational clause

$$\forall \vec{x}\,(t_1 \approx s_1 \vee \ldots \vee t_n \approx s_n \vee l_1 \not\approx r_1 \vee \ldots \vee l_k \not\approx r_k)$$

is valid iff

$$\exists \vec{x}\,(t_1 \not\approx s_1 \wedge \ldots \wedge t_n \not\approx s_n \wedge l_1 \approx r_1 \wedge \ldots \wedge l_k \approx r_k)$$

is unsatisfiable iff the Skolemized (ground!) formula

$$(t_1 \not\approx s_1 \wedge \ldots \wedge t_n \not\approx s_n \wedge l_1 \approx r_1 \wedge \ldots \wedge l_k \approx r_k)\{\vec{x} \mapsto \vec{c}\}$$

is unsatisfiable iff the formula

$$(t_1 \approx s_1 \vee \ldots \vee t_n \approx s_n \vee l_1 \not\approx r_1 \vee \ldots \vee l_k \not\approx r_k)\{\vec{x} \mapsto \vec{c}\}$$

is valid.

Flattening

$E = l_1 \approx r_1 \wedge \ldots \wedge l_n \approx r_n$

Flattening $E[f(t_1, \ldots, t_n)]_{p_1,\ldots,p_k} \Rightarrow_{\mathrm{CCF}}$
$E[c/p_1, \ldots, p_k] \wedge f(t_1, \ldots, t_n) \approx c$
provided all t_i are constants, the p_j are all positions in E of $f(t_1, \ldots, t_n)$, $|p_k| > 2$ for some k, or, $p_k = n.2$ and $E|_{m.1}$ is not a constant for some n, and c is fresh

As a result: only two kinds of equations left.
Term equations: $f(c_{i_1}, \ldots, c_{i_n}) \approx c_{i_0}$
Constant equations: $c_i \approx c_j$.

Congruence Closure

The congruence closure algorithm is presented as a set of abstract rewrite rules operating on a pair of equations E and a set of rules R, $(E; R)$, similar to Knuth-Bendix completion, Section 4.4.

$(E_0; R_0) \Rightarrow_{\mathrm{CC}} (E_1; R_1) \Rightarrow_{\mathrm{CC}} (E_2; R_2) \Rightarrow_{\mathrm{CC}} \cdots$

At the beginning, $E = E_0$ is a set of constant equations and $R = R_0$ is the set of term equations oriented from left-to-right. At termination, E is empty and R contains the result.

Simplify $(E \uplus \{c \doteq\approx c'\}; R \uplus \{c \to c''\}) \Rightarrow_{\mathrm{CC}}$
$(E \cup \{c'' \doteq\approx c'\}; R \cup \{c \to c''\})$

Delete $(E \uplus \{c \approx c\}; R) \Rightarrow_{\mathrm{CC}} (E; R)$

Orient $(E \uplus \{c \doteq\approx c'\}; R) \Rightarrow_{\mathrm{CC}} (E; R \cup \{c \to c'\})$
if $c \succ c'$

Deduce $(E; R \uplus \{t \to c,\, t \to c'\}) \Rightarrow_{\mathrm{CC}}$
$(E \cup \{c \approx c'\}; R \cup \{t \to c\})$

Collapse $(E; R \uplus \{t[c]_p \to c',\, c \to c''\}) \Rightarrow_{\mathrm{CC}}$
$(E; R \cup \{t[c'']_p \to c',\, c \to c''\})$
$p \neq \epsilon$

For rule Deduce, t is either a term of the form $f(c_1, \ldots, c_n)$ or a constant c_i.
For rule Collapse, t is always of the form $f(c_1, \ldots, c_n)$