2.6.6 Theorem (Resolution Termination)

If reduction rules are preferred over inference rules and no inference rule is applied twice to the same clause(s), then $\Rightarrow^+_{\text{RES}}$ is well-founded.
### The Overall Picture

<table>
<thead>
<tr>
<th>Application</th>
<th>System + Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>System</td>
<td>Algorithm + Implementation</td>
</tr>
<tr>
<td>Algorithm</td>
<td>Calculus + Strategy</td>
</tr>
<tr>
<td>Calculus</td>
<td>Logic + States + Rules</td>
</tr>
<tr>
<td>Logic</td>
<td>Syntax + Semantics</td>
</tr>
</tbody>
</table>
The CDCL calculus tests satisfiability of a finite set $N$ of propositional clauses.

I assume that $\bot \notin N$ and that the clauses in $N$ do not contain duplicate literal occurrences. Furthermore, duplicate literal occurrences are always silently removed during rule applications of the calculus. (Exhaustive Condensation.)
The CDCL calculus explicitly builds a candidate model for a clause set. If such a sequence of literals $L_1, \ldots, L_n$ satisfies the clause set $N$, it is done. If not, there is a false clause $C \in N$ with respect to $L_1, \ldots, L_n$.

Now instead of just backtracking through the literals $L_1, \ldots, L_n$, CDCL generates in addition a new clause, called learned clause via resolution, that actually guarantees that the subsequence of $L_1, \ldots, L_n$ that caused $C$ to be false will not be generated anymore.

This causes CDCL to be exponentially more powerful in proof length than its predecessor DPLL or Tableau.
A CDCL problem state is a five-tuple \((M; N; U; k; D)\) where

- \(M\) is a sequence of annotated literals, called a *trail*,
- \(N\) and \(U\) are sets of clauses,
- \(k \in \mathbb{N}\), and
- \(D\) is a non-empty clause or \(\top\) or \(\bot\), called the *mode* of the state.

The set \(N\) is initialized by the problem clauses where the set \(U\) contains all newly learned clauses that are consequences of clauses from \(N\) derived by resolution.
Modes of CDCL States

\((\epsilon; N; \emptyset; 0; \top)\) is the start state for some clause set \(N\)

\((M; N; U; k; \top)\) is a final state, if \(M \models N\) and all literals from \(N\) are defined in \(M\)

\((M; N; U; k; \bot)\) is a final state, where \(N\) has no model

\((M; N; U; k; D)\) is an intermediate model search state if \(M \not\models N\)

\((M; N; U; k; D)\) is a backtracking state if \(D \not\in \{\top, \bot\}\)
The Role of Levels

Literals in $L \in M$ are either annotated with a number, a level, i.e., they have the form $L^k$ meaning that $L$ is the $k^{th}$ guessed decision literal, or they are annotated with a clause that forced the literal to become true.

A literal $L$ is of level $k$ with respect to a problem state $(M; N; U; j; C)$ if $L$ or $\text{comp}(L)$ occurs in $M$ and $L$ itself or the first decision literal left from $L$ ($\text{comp}(L)$) in $M$ is annotated with $k$. If there is no such decision literal then $k = 0$.

A clause $D$ is of level $k$ with respect to a problem state $(M; N; U; j; C)$ if $k$ is the maximal level of a literal in $D$. 
CDCL Rules

**Propagate** \((M; N; U; k; \top) \Rightarrow_{\text{CDCL}} (ML^{C \lor L}; N; U; k; \top)\) provided \(C \lor L \in (N \cup U)\), \(M \models \neg C\), and \(L\) is undefined in \(M\)

**Decide** \((M; N; U; k; \top) \Rightarrow_{\text{CDCL}} (ML^{k+1}; N; U; k + 1; \top)\) provided \(L\) is undefined in \(M\)

**Conflict** \((M; N; U; k; \top) \Rightarrow_{\text{CDCL}} (M; N; U; k; D)\) provided \(D \in (N \cup U)\) and \(M \models \neg D\)
Skip \[(ML^{C\lor L}; N; U; k; D) \Rightarrow_{CDCL} (M; N; U; k; D)\]
provided \(D \not\in \{\top, \bot\}\) and \(\text{comp}(L)\) does not occur in \(D\)

Resolve \[(ML^{C\lor L}; N; U; k; D \lor \text{comp}(L)) \Rightarrow_{CDCL} (M; N; U; k; D \lor C)\]
provided \(D\) is of level \(k\)

Backtrack \[(M_1 K^{i+1} M_2; N; U; k; D \lor L) \Rightarrow_{CDCL} (M_1 L^{D\lor L}; N; U \cup \{D \lor L\}; i; \top)\]
provided \(L\) is of level \(k\) and \(D\) is of level \(i\).

Restart \[(M; N; U; k; \top) \Rightarrow_{CDCL} (\epsilon; N; U; 0; \top)\]
provided \(M \not\models N\)

Forget \[(M; N; U \cup \{C\}; k; \top) \Rightarrow_{CDCL} (M; N; U; k; \top)\]
provided \(M \not\models N\)
2.9.5 Definition (Reasonable CDCL Strategy)

A CDCL strategy is *reasonable* if the rules Conflict and Propagate are always preferred over all other rules.
2.9.6 Proposition (CDCL Basic Properties)

Consider CDCL run deriving \((M; N; U; k; C)\) by any strategy but without Restart and Forget. Then the following properties hold:

1. \(M\) is consistent.
2. All learned clauses are entailed by \(N\).
3. If \(C \notin \{\top, \bot\}\) then \(M \models \neg C\).
4. If \(C = \top\) and \(M\) contains only propagated literals then for each valuation \(\mathcal{A}\) with \(\mathcal{A} \models N\) it holds that \(\mathcal{A} \models M\).
5. If \(C = \bot\), \(M\) contains only propagated literals and \(M \models \neg D\) for some \(D \in (N \cup U)\) then \(N\) is unsatisfiable.
6. If \(C = \bot\) then CDCL terminates and \(N\) is unsatisfiable.
7. \(k\) is the maximal level of a literal in \(M\).
8. Each infinite derivation contains an infinite number of Backtrack applications.
2.9.7 Lemma (CDCL Redundancy)

Consider a CDCL derivation by a reasonable strategy. Then CDCL never learns a clause contained in $N \cup U$. 

November 5, 2020 65/91
2.9.10 Lemma (CDCL Soundness)

In a reasonable CDCL derivation, CDCL can only terminate in two different types of final states: $(M; N; U; k; \top)$ where $M \models N$ and $(M; N; U; k; \bot)$ where $N$ is unsatisfiable.
2.9.11 Proposition (CDCL Soundness)

The rules of the CDCL algorithm are sound: (i) if CDCL terminates from \((\epsilon; N; \emptyset; 0; \top)\) in the state \((M; N; U; k; \top)\), then \(N\) is satisfiable, (ii) if CDCL terminates from \((\epsilon; N; \emptyset; 0; \top)\) in the state \((M; N; U; k; \bot)\), then \(N\) is unsatisfiable.
2.9.12 Proposition (CDCL Strong Completeness)

The CDCL rule set is complete: for any valuation $M$ with $M \models N$ there is a reasonable sequence of rule applications generating $(M'; N; U; k; \top)$ as a final state, where $M$ and $M'$ only differ in the order of literals.
2.9.13 Proposition (CDCL Termination)

Assume the algorithm CDCL with all rules except Restart and Forget is applied using a reasonable strategy. Then it terminates in a state \((M; N; U; k; D)\) with \(D \in \{\top, \bot\}\).
## The Overall Picture

<table>
<thead>
<tr>
<th>Application</th>
<th>System + Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algorithm</td>
<td>System + Implementation</td>
</tr>
<tr>
<td>Calculus</td>
<td>Algorithm + Strategy</td>
</tr>
<tr>
<td>Logic</td>
<td>Calculus + States + Rules</td>
</tr>
<tr>
<td>Syntax</td>
<td>Logic + Syntax + Semantics</td>
</tr>
</tbody>
</table>
Algorithm: 5 CDCL($S$)

Input: An initial state $(\varepsilon; \mathcal{N}; \emptyset; 0; \top)$.

Output: A final state $S = (M; \mathcal{N}; U; k; \top)$ or $S = (M; \mathcal{N}; U; k; \bot)$

while (any rule applicable) do

3 if rule (Conflict($S$)) then

4 while (Skip($S$) || Resolve($S$)) do

5 update VSIDS on resolved literals;

6 update VSIDS on learned clause, Backtrack($S$);

7 if (forget heuristic) then

8 Forget($S$), Restart($S$);

9 else

10 if (restart heuristic) then

11 Restart($S$);

12 else

13 if rule (! Propagate($S$)) then

14 Decide($S$) literal with max. VSIDS score;

15 return ($S$);