3.13.7 Lemma (Lifting)

Let $D \vee L$ and $C \vee L'$ be variable-disjoint clauses and σ a grounding substitution for $C \vee L$ and $D \vee L'$. If there is a superposition left inference
$(N \uplus \{(D \vee L)\sigma, (C \vee L')\sigma\}) \Rightarrow_{\text{SUP}}$
$(N \cup \{(D \vee L)\sigma, (C \vee L')\sigma\} \cup \{D\sigma \vee C\sigma\})$ and if
$\text{sel}((D \vee L)\sigma) = \text{sel}((D \vee L))\sigma$, $\text{sel}((C \vee L')\sigma) = \text{sel}((C \vee L'))\sigma$, then there exists a mgu τ such that
$(N \uplus \{D \vee L, C \vee L'\}) \Rightarrow_{\text{SUP}} (N \cup \{D \vee L, C \vee L'\} \cup \{(D \vee C)\tau\})$.

Let $C \vee L \vee L'$ be a clause and σ a grounding substitution for $C \vee L \vee L'$. If there is a factoring inference
$(N \uplus \{(C \vee L \vee L')\sigma\}) \Rightarrow_{\text{SUP}} (N \cup \{(C \vee L \vee L')\sigma\} \cup \{(C \vee L)\sigma\})$
and if $\text{sel}((C \vee L \vee L')\sigma) = \text{sel}((C \vee L \vee L'))\sigma$, then there exists a mgu τ such that
$(N \uplus \{C \vee L \vee L'\}) \Rightarrow_{\text{SUP}} (N \cup \{C \vee L \vee L'\} \cup \{(C \vee L)\tau\})$

3.13.8 Example (First-Order Reductions are not Liftable)

Consider the two clauses $P(x) \vee Q(x)$, $P(g(y))$ and grounding substitution $\{x \mapsto g(a), y \mapsto a\}$. Then $P(g(y))\sigma$ subsumes $(P(x) \vee Q(x))\sigma$ but $P(g(y))$ does not subsume $P(x) \vee Q(x)$. For all other reduction rules similar examples can be constructed.

3.13.9 Lemma (Soundness and Completeness)

First-Order Superposition is sound and complete.

3.13.10 Lemma (Redundant Clauses are Obsolete)

If a clause set N is unsatisfiable, then there is a derivation $N \Rightarrow^*_{\mathrm{SUP}} N'$ such that $\bot \in N'$ and no clause in the derivation of $\bot$ is redundant.

3.13.11 Lemma (Model Property)

If N is a saturated clause set and $\bot \notin N$ then $\mathrm{grd}(\Sigma, N)_{\mathcal{I}} \models N$.

Decision Procedures for BS

3.15.3 Definition (Bernays-Schoenfinkel Fragment (BS))

A formula of the Bernays-Schoenfinkel fragment has the form $\exists\vec{x}.\forall\vec{y}.\phi$ such that ϕ does not contain quantifiers nor non-constant function symbols.

3.15.4 Theorem (BS is decidable)

Unsatisfiability of a BS clause set is decidable.

$$1 : \neg R(x, y) \lor \neg R(y, z) \lor R(x, z)$$

$$2 : R(x, y) \lor R(y, x)$$

A state is now a set of clause sets. Let k be the number of different constants $a_1, \ldots, a_k$ in the initial clause set N. Then the initial state is the set $M = \{N\}$, Superposition Left is adopted to the new setting, Factoring is no longer needed and the rules Instantiate and Split are added. The variables $x_1, \ldots, x_k$ constitute a *variable chain* between literals L_1, L_k inside a clause C, if there are literals $\{L_1, \ldots, L_k\} \subseteq C$ such that $x_i \in (\mathrm{vars}(L_i) \cap \mathrm{vars}(L_{i+1}))$, $1 \leq i < k$.

Superposition BS

$M \uplus \{N \uplus \{P(t_1, \ldots, t_n), C \vee \neg P(s_1, \ldots, s_n)\}\} \;\Rightarrow_{\text{SUPBS}}$
$M \cup \{N \cup \{P(t_1, \ldots, t_n), C \vee \neg P(s_1, \ldots, s_n)\} \cup \{C\sigma\}\}$

where (i) $\neg P(s_1, \ldots, s_n)$ is selected in $(C \vee \neg P(s_1, \ldots, s_n))\sigma$ (ii) σ is the mgu of $P(t_1, \ldots, t_n)$ and $P(s_1, \ldots, s_n)$
(iii) $C \vee \neg P(s_1, \ldots, s_n)$ is a Horn clause

Instantiation $M \uplus \{N \uplus \{C \vee A_1 \vee A_2\}\} \;\Rightarrow_{\text{SUPBS}}$
$M \cup \{N \cup \{(C \vee A_1 \vee A_2)\sigma_i \mid \sigma_i = \{x \mapsto a_i\}, 1 \leq i \leq k\}\}$

where x occurs in a variable chain between A_1 and A_2

Split $M \uplus \{N \uplus \{C_1 \vee A_1 \vee C_2 \vee A_2\}\}$
$\Rightarrow_{\text{SUPBS}} \; M \cup \{N \cup \{C_1 \vee A_1\}, N \cup \{C_2 \vee A_2\}\}$

where $\text{vars}(C_1 \vee A_1) \cap \text{vars}(C_2 \vee A_2) = \emptyset$

3.16.1 Definition (Rigorous Selection Strategy)

A selection strategy is *rigorous* of in any clause containing a negative literal, a negative literal is selected.

3.16.2 Lemma (SUPBS Basic Properties)

The SUPBS rules have the following properties:

1. Superposition BS is sound.
2. Instantiation is sound and complete.
3. Split is sound and complete.

Alternative Condensation Rule

The Condensation-BS rule turns Superposition (Resolution) into a decision procedure for the Bernays-Schönfinkel fragment and is an alternative to the SUPBS calculus.

Condensation-BS $(N \uplus \{L_1 \vee \cdots \vee L_n\}) \;\Rightarrow_{\text{SUP}}$
$(N \cup \{\text{rdup}((L_1 \vee \ldots L_n)\sigma_{i,j}) \mid \sigma_{i,j} = \text{mgu}(L_i, L_j)$ and $\sigma_{i,j} \neq \bot\})$
provided any ground instance $(L_1 \vee \cdots \vee L_n)\delta$ contains at least two duplicate literals