Decidable First-Order (Clause) Classes

(i) Shallow Linear Monadic Horn Clauses
   [Weidenbach, 1999, CADE]

(ii) Guarded Fragment
     [Harald Ganzinger and Hans de Nivelle, 1999, LICS]

(iii) Monadic Fragment with Equality
      [Leo Bachmair and Harald Ganzinger and Uwe Waldmann, 1993, Computational Logic and Proof Theory]
Idea of the Decidability Proofs

Show that there are only finitely many superposition inferences with respect to redundancy.

More concretely: show that for any (derived) clause $C$ both the number of variables and the maximal depth of terms can be bound. Then Subsumption and Condensation guarantee termination.
Subsumption \[(N \uplus \{C_1, C_2\}) \Rightarrow_{\text{RES}} (N \uplus \{C_1\})\]
provided \(C_1\sigma \subset C_2\) for some matcher \(\sigma\)

Tautology Deletion \[(N \uplus \{C \lor A \lor \neg A\}) \Rightarrow_{\text{RES}} (N)\]

Condensation \[(N \uplus \{C\}) \Rightarrow_{\text{RES}} (N \uplus \{C'\})\]
where \(C'\) is the result of removing duplicate literals from \(C\sigma\) for some matcher \(\sigma\) and \(C'\) subsumes \(C\)
Clauses are of the form

$$\neg A_1 \lor \ldots \lor \neg A_n \lor B$$

where

(i) clauses may be purely negative (without $B$) or purely positive (just $B$)

(ii) $B$ has the form $S(x)$, or $S(c)$, or $S(f(x_1, \ldots, x_m))$ all $x_i$ different

(iii) $A_i$ has the form $S(t)$ for an arbitrary term $t$
Guarded Fragment

Guarded Fragment (GF) Definition

Recursively defined by the following rules (no equality, no function symbols):

(i) Every atom $A$ is in GF

(ii) If $\phi, \psi$ are in GF so are $\neg \phi$, $\phi \circ \psi$ for $\circ \in \{\land, \lor, \rightarrow, \leftrightarrow\}$

(iii) If $\phi$ in GF and $A$ an atom such that every free variable of $\phi$ occurs among the arguments of $A$, then $\forall \vec{x}.(A \rightarrow \phi)$ and $\exists \vec{x}.(A \land \phi)$ are in GF.
A clause $C$ belongs to the CNF of a formula $\phi$ from GF iff

(i) Every non-ground functional term from $C$ contains all variables if $C$

(ii) If $C$ is not ground, then there exists a negative literal $\neg A \in C$ that does not contain a non-ground, functional term but all variables of $C$. 
Monadic Fragment with Equality
(Löwenheim 1915)

Originally, formulas of the form

$$\{\forall, \exists\}^* \phi$$

and $\phi$ is quantifier free, no function symbols, only monadic predicates.

Building a CNF we get flat clauses where

(i) all atoms are of the form $S(t)$ or $s \approx t$

(ii) there exists a sequence of distinct variables auch that
any term $t$ is either a variable $x_n$ or of the form
$f(x_1, \ldots, x_n)$ for $n \leq m$. 
Using an LPO one can guarantee that superposition inferences between flat clauses again result in flat clauses.

Furthermore we add the rule Split where now a superposition state consists of a set $M$ of clause sets:

$\text{Split} \quad M \cup \{N \cup \{C_1 \lor A_1 \lor C_2 \lor A_2\}\}$

$\Rightarrow \text{SUP} \quad M \cup \{N \cup \{C_1 \lor A_1\}, N \cup \{C_2 \lor A_2\}\}$

where $\text{vars}(C_1 \lor A_1) \cap \text{vars}(C_2 \lor A_2) = \emptyset$