Cooper [16] showed that given two strict inequations $x < t$ and $x > s$, and a divisibility constraint $d \mid x$ the variable x can be eliminated in the above described way.

$$\exists x.(x < t \wedge x > s \wedge d \mid x) \quad \text{iff} \quad \bigvee_{i=1}^{i \leq d} (s + i < t \wedge d \mid s + i)$$

This needs to be further generalized to cope with $\nmid$, multiple inequations, and divisibility constraints for some variable x. The actual procedure is then similar to virtual substitution, Section 6.2.3. Note that virtual substitution was invented after Cooper's algorithm for variable elimination over the integers.

Let $\exists x.\phi$ be a formula of LIA, where ϕ is in negation normal form, ϕ does not contain any quantifiers nor negation symbols, and the LIA relations occurring in ϕ are $\{<, >, \mid, \nmid\}$. Any LIA formula can be effectively transformed into this form, see the discussion above and Section 6.2.1, including the rule ElimNeg. Furthermore, for all inequations $cx \circ t$ and divisibility atoms $a \circ' bx + s$, $\circ \in \{<, >\}$, $\circ' \in \{\mid, \nmid\}$, I assume $c = 1$, $b = 1$.

If c is negative for some inequation it is multiplied by -1 and then transformed into its strict form. If b is negative, for divisibility atoms it is sufficient to multiply the right hand side by -1.

If there are atoms

$$\begin{array}{rcl} c_i x & \circ_i & t_i \\ a_j & \circ'_j & b_j x + s_j \end{array}$$

in ϕ with $c_i > 1$ or $b_j > 1$ for some i, j, $\circ_i \in \{<, >\}$, $\circ'_j \in \{\mid, \nmid\}$, then the lcm d of the c_i, b_j is computed. The atoms are first replaced by

$$\begin{array}{rcl} dx & \circ_i & \frac{d}{c_i} t_i \\ \frac{d}{b_j} a_j & \circ'_j & dx + \frac{d}{b_j} s_j \end{array}$$

respectively, and finally they are replaced by

$$\begin{array}{rcl} x & \circ_i & \frac{d}{c_i} t_i \\ \frac{d}{b_j} a_j & \circ'_j & x + \frac{d}{b_j} s_j \\ d & \mid & x \end{array}$$

respectively, where the divisibility constraint $d \mid x$ is added conjunctively to ϕ.

Similar to the arguments for composing the virtual substitution test points, solutions for $\exists x.\phi$ can be considered from $-\infty$ to ∞ or the other way round. I explain the former, the latter is then a standard exercise. Let $x < t_i$, $x > s_j$, $a_k \mid x + r_k$, $b_h \nmid x + l_h$ be all atoms in ϕ containing x where the t_i, s_j, r_k, l_h do not contain x. Let $p_1, \ldots, p_n$ be the positions of the atoms $x < t_i$ in ϕ and $q_1, \ldots, q_o$ be the positions of the atoms $x > s_j$ in ϕ. Let d be the lcm of the a_k, b_h. Then