First-Order Logic with Equality

In this Chapter I combine the ideas of Superposition for first-order logic without equality, Section 3.13, and Knuth-Bendix Completion, Section 4.4, to get a calculus for equational clauses.

Recall that predicative literals can be translated into equations

$$\begin{array}{lcl} P(t_1, \dots, t_n) & \Rightarrow & f_P(t_1, \dots, t_n) \approx \text{true} \\ \neg P(t_1, \dots, t_n) & \Rightarrow & f_P(t_1, \dots, t_n) \not\approx \text{true} \end{array}$$

The ground inference rules corresponding to Knuth-Bendix critical pair computation generalized to clauses and Superposition Left on first-order logic wihtout equality modulo a reduction ordering $\succ$ that is total on ground terms. Then the construction of Definition 3.12.1 is lifted to equational clauses.

The multiset $\{s, t\}$ is assigned to a positive literal $s \approx t$, the multiset $\{s, s, t, t\}$ is assigned to a negative literal $s \not\approx t$. The *literal ordering* $\succ_L$ compares these multisets using the multiset extension of $\succ$. The *clause ordering* $\succ_C$ compares clauses by comparing their multisets of literals using the multiset extension of $\succ_L$. Eventually $\succ$ is used for all three orderings depending on the context.

Superposition Left

$(N \uplus \{D \vee t \approx t', C \vee s[t] \not\approx s'\}) \;\Rightarrow$
$(N \cup \{D \vee t \approx t', C \vee s[t] \not\approx s'\} \cup \{D \vee C \vee s[t'] \not\approx s'\})$

where $t \approx t'$ is strictly maximal and $s \approx s'$ are maximal in their respective clauses, $t \succ t'$, $s \succ s'$

Superposition Right

$(N \uplus \{D \vee t \approx t', C \vee s[t] \approx s'\}) \;\Rightarrow$
$(N \cup \{D \vee t \approx t', C \vee s[t] \approx s'\} \cup \{D \vee C \vee s[t'] \approx s'\})$

where $t \approx t'$ and $s \approx s'$ are strictly maximal in their respective clauses, $t \succ t'$, $s \succ s'$

Equality Resolution $(N \uplus \{C \vee s \not\approx s\}) \;\Rightarrow\; (N \cup \{C \vee s \not\approx s\} \cup \{C\})$

where $s \not\approx s$ is maximal in the clause

Factoring is more complicated due to more complicated partial models. Classical Herbrand interpretation not sufficient because of equality.

The solution is to define a set E of ground equations and take $T(\Sigma, \emptyset)/E = T(\Sigma, \emptyset)/{\approx_E}$ as the universe. Then two ground terms s and t are equal in the interpretation if and only if $s \approx_E t$. If E is a terminating and confluent rewrite system R, then two ground terms s and t are equal in the interpretation, if and only if $s \downarrow_R t$.

Now the problem with the standard factoring rule is that in the completeness proof for the superposition calculus without equality, the following property holds: if $C = C' \vee A$ with a strictly maximal atom A is false in the current interpretation N_C with respect to some clause set, see Definition 3.12.5, then adding A to the current interpretation cannot make any literal in C' true.

This does not hold anymore in the presence of equality. Let $b \succ c \succ d$. Assume that the current rewrite system (representing the current interpretation) contains the rule $c \to d$. Now consider the clause $b \approx c \vee b \approx d$.

Equality Factoring $(N \uplus \{C \vee s \approx t' \vee s \approx t\}) \Rightarrow (N \cup \{C \vee s \approx t' \vee s \approx t\} \cup \{C \vee t \not\approx t' \vee s \approx t'\})$

where $s \succ t'$, $s \succ t$ and $s \approx t$ is maximal in the clause

The lifting from the ground case to the first-order case with variables is then identical to the case of superposition without equality: identity is replaced by unifiability, the mgu is applied to the resulting clause, and $\succ$ is replaced by $\not\preceq$.
An addition, as in Knuth-Bendix completion, overlaps at or below a variable position are not considered. The consequence is that there are inferences between ground instances $D\sigma$ and $C\sigma$ of clauses D and C which are not ground instances of inferences between D and C. Such inferences have to be treated in a special way in the completeness proof and will be shown to be obsolete.

Superposition Right

$(N \uplus \{D \lor t \approx t', C \lor s[u] \approx s'\}) \Rightarrow$
$(N \cup \{D \lor t \approx t', C \lor s[u] \approx s'\} \cup \{(D \lor C \lor s[t'] \approx s')\sigma\})$

where σ is the mgu of t, u, u is not a variable $t\sigma \not\preceq t'\sigma$, $s\sigma \not\preceq s'\sigma$, $(t \approx t')\sigma$ strictly maximal in $(D \lor t \approx t')\sigma$, nothing selected and $(s \approx s')\sigma$ maximal in $(C \lor s \approx s')\sigma$ and nothing selected

Superposition Left

$(N \uplus \{D \lor t \approx t', C \lor s[u] \not\approx s'\}) \Rightarrow$
$(N \cup \{D \lor t \approx t', C \lor s[u] \not\approx s'\} \cup \{(D \lor C \lor s[t'] \not\approx s')\sigma\})$

where σ is the mgu of t, u, u is not a variable $t\sigma \not\preceq t'\sigma$, $s\sigma \not\preceq s'\sigma$, $(t \approx t')\sigma$ strictly maximal in $(D \lor t \approx t')\sigma$, nothing selected and $(s \not\approx s')\sigma$ maximal in $(C \lor s \not\approx s')\sigma$ or selected

Equality Resolution $(N \uplus \{C \vee s \not\approx s'\}) \ \Rightarrow (N \cup \{C \vee s \not\approx s'\} \cup \{C\sigma\})$

where σ is the mgu of s, s', $(s \not\approx s')\sigma$ maximal in $(C \vee s \not\approx s')\sigma$ or selected

Equality Factoring $(N \uplus \{C \vee s' \approx t' \vee s \approx t\}) \ \Rightarrow (N \cup \{C \vee s' \approx t' \vee s \approx t\} \cup \{(C \vee t \not\approx t' \vee s \approx t')\sigma\})$

where σ is the mgu of s, s', $s'\sigma \not\preceq t'\sigma$, $s\sigma \not\preceq t\sigma$, $(s \approx t)\sigma$ maximal in $(C \vee s' \approx t' \vee s \approx t)\sigma$ and nothing selected

5.2.1 Theorem (Superposition Soundness)

All inference rules of the superposition calculus are *sound*, i.e., for every rule $N \uplus \{C_1, \ldots, C_n\} \Rightarrow N \cup \{C_1, \ldots, C_n\} \cup \{D\}$ it holds that $\{C_1, \ldots, C_n\} \models D$.

5.2.2 Definition (Abstract Redundancy)

A clause C is *redundant* with respect to a clause set N if for all ground instances $C\sigma$ there are clauses $\{C_1, \ldots, C_n\} \subseteq N$ with ground instances $C_1\tau_1, \ldots, C_n\tau_n$ such that $C_i\tau_i \prec C\sigma$ for all i and $C_1\tau_1, \ldots, C_n\tau_n \models C\sigma$.

5.2.3 Definition (Partial Model Construction)

Given a clause set N and an ordering $\succ$ a (partial) model $N_{\mathcal{I}}$ can be constructed inductively over all ground clause instances of N as follows:

$$N_C := \bigcup_{D \prec C}^{D \in \text{ground}(\Sigma, N)} E_D$$

$$N_{\mathcal{I}} := \bigcup_{C \in \text{ground}(\Sigma, N)} N_C$$

where N_D, $N_{\mathcal{I}}$, E_D are also considered as rewrite systems with respect to $\succ$. If $E_D \neq \emptyset$ then D is called *productive*.

$$E_D := \begin{cases} \{s \approx t\} & \text{if } D = D' \vee s \approx t, \\ & (i)\ s \approx t \text{ is strictly maximal in } D \\ & (ii)\ s \succ t \\ & (iii)\ D \text{ is false in } N_D \\ & (iv)\ D' \text{ is false in } N_D \cup \{s \rightarrow t\} \\ & (v)\ s \text{ is irreducible by } N_D \\ & (vi) \text{ no negative literal is selected in } D' \\ \emptyset & \text{otherwise} \end{cases}$$