Free-Variable Tableau

An important disadvantage of standard first-order tableau is that the γ ground term instances need to be guessed. The main complexity in proving a formula to be valid lies in this guessing as for otherwise tableau terminates with a proof. Guessing useless ground terms may result in infinite branches.

A natural idea is to guess ground terms that can eventually be used to close a branch. Of course, it is not known which ground term will close a branch. Therefore, it would be great to postpone the γ instantiations. This is the idea of free-variable first-order tableau.

Instead of guessing a ground term for a γ formula, free-variable tableau introduces a fresh variable. Then a branch can be closed if two complementary literals have a common ground instance, i.e., their atoms are unifiable. The instantiation is delayed until a branch is closed for two literals via unification. As a consequence, for δ formulas no longer constants are introduced but shallow, so called *Skolem* terms in the formerly universally quantified variables that had the δ formula in their scope.

The new calculus needs to keep track of scopes of variables, so I move from a state as a set of pairs of a sequence and a set of constants, to a set of sequences of pairs (M_i, X_i) where X_i is a set of variables.

3.8.1 Definition (Direct Free-Variable Tableau Descendant)

Given a γ- or δ-formula ϕ its direct descendants are

γ	Descendant $\gamma(y)$
$\forall x_S.\psi$	$\psi\{x_S \mapsto y_S\}$
$\neg\exists x_S.\psi$	$\neg\psi\{x_S \mapsto y_S\}$
	for a fresh variable y_S

δ	Descendant $\delta(f(y_1,\ldots,y_n))$
$\exists x_S.\psi$	$\psi\{x_S \mapsto f(y_1,\ldots,y_n)\}$
$\neg\forall x_S.\psi$	$\neg\psi\{x_S \mapsto f(y_1,\ldots,y_n)\}$
	for some fresh Skolem function f,
	$f : \text{sort}(y_1) \times \ldots \times \text{sort}(y_n) \to S$

The notion of closedness transfers exactly from standard to free-variable tableau.

For α- and β-formulas the definition of an *open* formula remains unchanged as well. A γ- or δ-formula is called *open* in (M, X) if no direct descendant is contained in M. Note that instantiation of a tableau may remove direct descendants of γ- or δ-formulas by substituting terms for variables.

Then a branch, pair (M, X), sequence M, is *open* if it is not closed and there is an open formula in M or there is pair of unifiable, complementary literals in M.

γ-Expansion $N \uplus \{((\phi_1, \ldots, \psi, \ldots, \phi_n), X)\} \Rightarrow_{FT}$
$N \cup \{((\phi_1, \ldots, \psi, \ldots, \phi_n, \psi'), X \cup \{y\})\}$
provided ψ is a γ-formula, ψ' a $\gamma(y)$ descendant where y is fresh to the overall tableau and the sequence is not closed.

δ-Expansion $N \uplus \{((\phi_1, \ldots, \psi, \ldots, \phi_n), X)\} \Rightarrow_{FT}$
$N \cup \{(\phi_1, \ldots, \psi, \ldots, \phi_n, \psi'), X)\}$
provided ψ is an open δ-formula, $X = \{y_1, \ldots, y_n\}$, ψ' a $\delta(f(y_1, \ldots, y_n))$ descendant where f is fresh to the sequence, and the sequence is not closed.

Branch-Closing $N \uplus \{((\phi_1, \ldots, \phi_n), X)\} \Rightarrow_{\mathrm{FT}}$ $(N \cup \{((\phi_1, \ldots, \phi_n), X)\})\sigma$

provided there are complementary literals ϕ_i and ϕ_j, $\mathrm{atom}(\phi_i)\sigma = \mathrm{atom}(\phi_j)\sigma$ for an mgu σ, and the sequence is not closed.

The first-order free-variable tableau calculus consists of the rules α-, and β-expansion from standard tableau which are adapted to pairs of sequences and variable sets, and the above three rules γ-Expansion, δ-Expansion and Branch-Closing.

It remains to define the instantiation of a tableau by a substitution. As usual the application of a substitution to a set means application to the elements. For a pair $((\phi_1, \ldots, \phi_n), X)$ it is defined by $((\phi_1, \ldots, \phi_n), X)\sigma := ((\phi_1\sigma, \ldots, \phi_n\sigma), X \setminus \mathrm{dom}(\sigma))$.

A possibly infinite tableau derivation $s_0 \Rightarrow_{\text{FT}} s_1 \Rightarrow_{\text{FT}} \ldots$ is called *saturated* if for all its open sequences M_i of some pair $(M_i, X_i) \in s_i$ and all formulas ϕ occurring in M_i, there is an index $j > i$ and some pair $(M_j, X_j) \in s_j$, M_i is a prefix of M_j, if in case ϕ is an α-formula then both direct descendants are part of M_j, if it is a β-formula then one of its descendants is part of M_j, if it is a δ- or γ-formula then one direct descendant is part of M_j, and if Branch-Closing is applicable to M_i then M_j is closed.

3.8.2 Theorem (Free-variable First-Order Tableau is Sound and Complete)

A formula ϕ is valid iff free-variable tableau computes a closed state out of $\{(\neg\phi, \emptyset)\}$.